Derivative of the Exponential Function With Base e

Objectives: Find the derivative of exponential functions.

Estimations of *e*

If you began walking at 1 km/h and then doubled your speed over a one-minute interval, you would be walking at 2 km/h. But suppose you increased your speed by 50% every half-minute. How fast would you be walking at the end of one minute?

$1 + 0.50(1) = 1.5$ $1.5 + 0.5(1.5) = 2.25$ $1(1.50)^2 = 2.25$

Suppose you increased your speed by 25% every quarter-minute (15 seconds). What would your speed be at the end of one minute? Remember, your speed would be 1.25 times as fast every quarter-minute. Complete the chart below

$1 + 1(0.25)$

$1.25 + 0.25(1.25)$

$1.5625 + 0.25(1.5625)$

Time Elapsed (s)	0	15	30	45	60
Speed (km/h)	1	1.25	1.5625	1.95...	2.44...

$+ 0.25(1.95...)$

Generate an expression to find your speed at the end of one minute.

$$\text{speed} = 1(1.25)^4 = 2.44...$$

Suppose you increased your speed by $\frac{1}{10}$ for every tenth of a minute. What would your speed be at the end of one minute?

$$\text{speed} = 1\left(1 + \frac{1}{10}\right)^{10} = 1(1.10)^{10} = 2.59...$$

Complete the table for each increase in speed for and equal portion of a minute.

	Increase in speed	**Speed at the end of 1 minute**
	$\frac{1}{10}$ $s = 1\left(1 + \frac{1}{10}\right)^{10}$	2.5937424...
	$\frac{1}{1000}$ $s = \left(1 + \frac{1}{1000}\right)^{1000}$	2.7169239...
one hundred thousand increases...	$\frac{1}{100000}$ $s = \left(1 + \frac{1}{100\,000}\right)^{100\,000}$	2.718268...
ten million increases...	$\frac{1}{10000000}$	2.71828169...
one billion increases...	$\frac{1}{1000000000}$	2.718282...

Derivatives of $y = e^{x}$

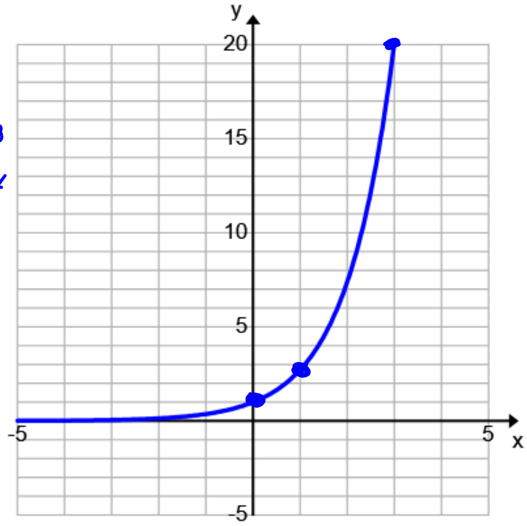
e can be defined as: $e = \lim_{n\to\infty}\left[1+\frac{1}{n}\right]^{n}$ $\approx 2.718281828...$

$y = e^x$

x	e^x
0	1
1	2.718
3	20.08
5	148.4

Investigate:
Why is *e* such a special number?

Use your calculator to sketch $y = e^{x}$.
Find the values of e^x at $x = 1,3,5$



Find the derivative of $y = e^{x}$ at $x = 1,3,5$ using your calculator.

P(0,1)
P(1, 2.718)
P(3, 20.08)

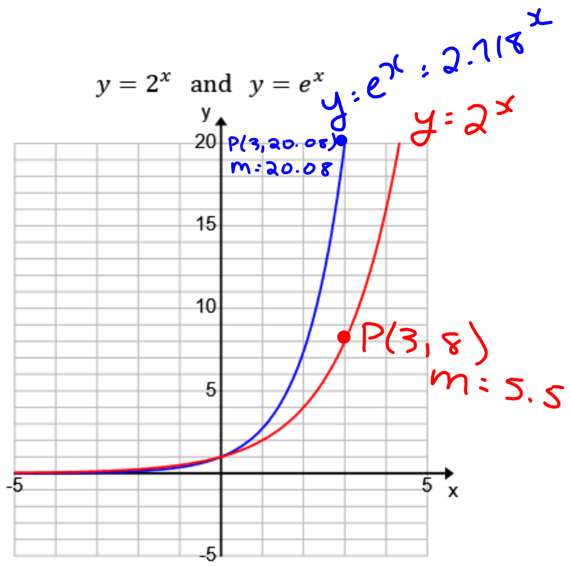
x	$y' = m$
0	1
1	2.718
3	20.08
5	148.4

State the value of $\frac{dy}{dx}e^x$

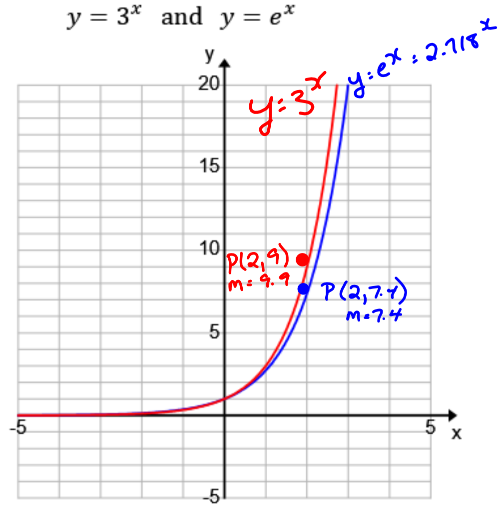
$\frac{d}{dx}(e^x) = e^x$

$y = e^x$ and $\frac{d}{dx}(e^x)$ are same function

$y = 2^x$ and $y = e^x$



$y = 3^x$ and $y = e^x$



Chain Rule: $\boldsymbol{f(x) = e^{u}}$ then $\boldsymbol{f'(x) = e^{u} \cdot \frac{du}{dx}}$

1. Differentiate

a) $y = x^3 e^x$

$f: x^3 \quad g: e^x$

$f': 3x^2 \quad g': e^x$

$\frac{dy}{dx} = (3x^2)(e^x) + (x^3)(e^x)$

$= x^2 e^x (3 + x)$

b) $y = e^{x^2}$

$\frac{dy}{dx} = (e^{x^2}) \frac{d}{dx}(x^2)$

$\frac{dy}{dx} = 2x e^{x^2}$

c) $y = x^5 e^{x^5}$

$f: x^5 \quad g = e^{x^5}$

$f': 5x^4 \quad g': e^{x^5} \cdot \frac{d}{dx}(x^5)$

$g': 5x^4 e^{x^5}$

$y': 5x^4 e^{x^5} + x^5(5x^4 e^{x^5})$

$y': 5x^4 e^{x^5}(1 + x^5)$

2. Find the absolute maximum value of the function $f(x) = xe^{-x}$ using the first or second derivative test.

$f: x \quad g: e^{-x}$

$f': 1 \quad g': (e^{-x}) \frac{d}{dx}(-x)$

$g': -e^{-x}$

$f'(x) = (1)(e^{-x}) + (x)(-e^{-x})$

$f'(x) = e^{-x}(1 - x)$

$0 = \frac{1 - x}{e^x}$

CN: $x - 1 = 0$

$x = 1$

$e^x \neq 0 \quad e^x > 0$

	$1-x$	e^x	f' slopes	$f(x)$
$(-\infty, 1)$	+	+	pos	INC
$(1, \infty)$	−	+	neg	DEC

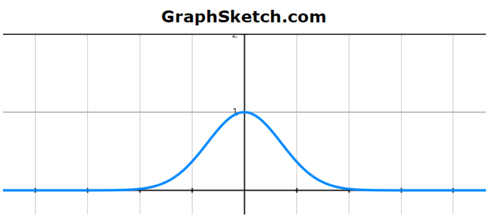
> max $f(1)$

$f(1) = (1)(e^{-1}) = \frac{1}{e}$

3. Given: $f(x) = e^{-x^2}$

a) Sketch the function.

b) Differentiate the function.



$f'(x) = [e^{-x^2}] \frac{d}{dx}(-x^2)$

$f'(x) = \frac{-2x}{e^{x^2}}$

Homework: Page 366 # 1, 4 (a,b,d,g,h,k,l) , 5, 8, 10, 11(a,b)

1. Simplify.

(a) $\dfrac{2}{e^{-x}}$ (b) $(e^x)^4$

(c) $e^{1-x}e^{3x}$ (d) $e^x e^{-x}$

(e) $e^{2x}(1 - 5e^{3x})$ (f) $\dfrac{6e^{8x}}{e^{3x}}$

4. Differentiate.

(a) $y = 2e^{-x}$ (b) $y = x^4e^x$

(c) $y = e^{2x}\sin 3x$ (d) $y = e^{\sqrt{x}}$

(e) $y = e^{\tan x}$ (f) $y = \tan(e^x)$

(g) $y = \dfrac{e^x}{x}$ (h) $y = \dfrac{e^x}{1 - e^{2x}}$

(i) $y = e^{\sin(x^2)}$ (j) $y = xe^{\cot 4x}$

(k) $y = (1 + 5e^{-10x})^4$ (l) $y = \sqrt{x + e^{1-x^2}}$

5. Find the equation of the tangent line to the curve $y = 1 + xe^{2x}$ at the point where $x = 0$.

8. Find the intervals of increase and decrease for the function $f(x) = x^2e^{-x}$.

9. Find the absolute minimum value of the function $f(x) = \dfrac{e^x}{x}$, $x > 0$.

10. For the function $f(x) = xe^x$, find

(a) the absolute minimum value,

(b) the intervals of concavity,

(c) the inflection point.

11. Evaluate.

(a) $\lim\limits_{x\to\infty} e^{-x}$ (b) $\lim\limits_{x\to-\infty} e^{-x}$ (c) $\lim\limits_{t\to\frac{\pi}{2}^+} e^{\tan t}$