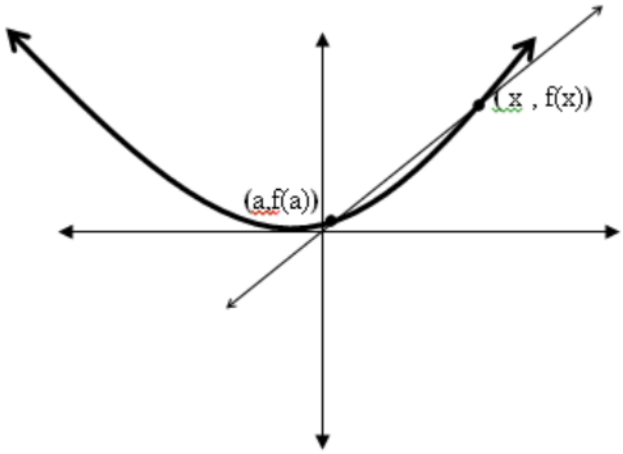
3.0 Limits for tangents

Outcome: Find tangent slopes using limits.

Method 1: Find the slope between two points $P_1(x, f(x))$ **and** $P_2(a, f(a))$



$$2\ points\ slope:\ \ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a) - f(x)}{a - x}$$

Develop a formula for finding slope of a line tangent to a curve at $P_1(x, f(x))$

Make the second point approach the given point:

$$\boldsymbol{tangent\ slope = m = \lim_{a \to x} \frac{f(a) - f(x)}{a - x}}$$

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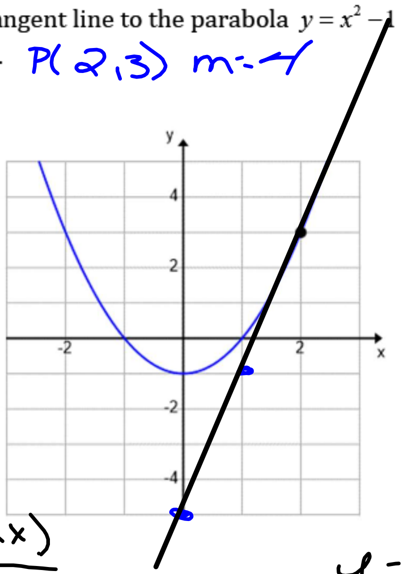
1. Given the function $f(x) = x^2 - 1$:
   a) Find the slope of the tangent line to the parabola $y = x^2 - 1$ at the point (2,3).
   b) Graph the tangent line.

$P(2,3)$ $m=4$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



$$m = \lim_{a \to x} \frac{f(a) - f(x)}{a - x}$$

$$= \lim_{a \to 2} \frac{(a^2 - 1) - (3)}{a - 2}$$

$$= \lim_{a \to 2} \frac{a^2 - 4}{a - 2}$$

$$= \lim_{a \to 2} \frac{\cancel{(a-2)}(a+2)}{\cancel{(a-2)}}$$

$y = x^2 - 1$

$f(a) = a^2 - 1$

$f(x) = x^2 - 1$

$f(2) = 3$

$$m = \lim_{a \to 2} (a+2)$$

$m = 2 + 2$

$m = 4$

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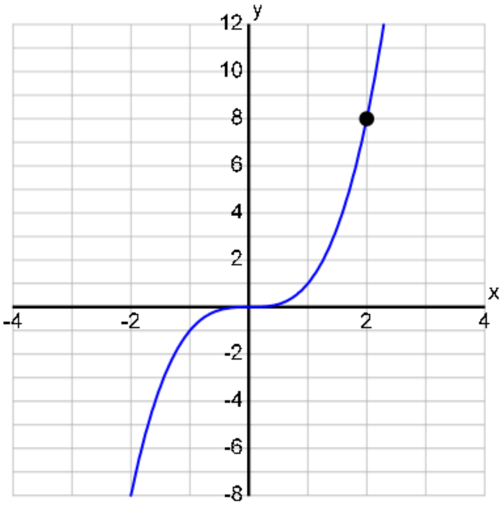
2. Find the slope and the equation of a tangent line to the cubic function $y = x^3$ at the point (2,8).



$m = \frac{y_2 - y_1}{x_2 - x_1}$

$P(2, 8)$

$P(a, a^3)$

$f(a) = a^3$

$m = \lim_{a \to 2} \frac{a^3 - 8}{a - 2}$ ... $(a-2)(a^2+2a+4)$

$= \lim_{a \to 2} \frac{\cancel{(a-2)}(a^2+2a+4)}{\cancel{(a-2)}}$

$= \lim_{a \to 2} a^2+2a+4$

$m = (2)^2 + 2(2) + 4$

$\boxed{m = 12}$ Point $(2, 8)$

$\frac{12}{1} = \frac{y-8}{x-2}$

$12x - 24 = y - 8$

$12x - y = 16$ OR $12x - y - 16 = 0$

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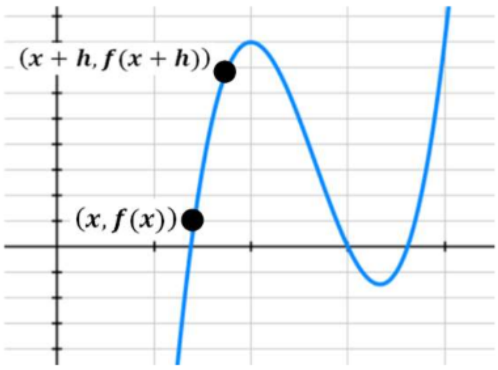
Method 2:

Develop a formula for finding slope of a line tangent to a curve at $P_1(x, f(x))$ using First Principle of Calculus.

Find the slope between two points $P_1(x, f(x))$ **and** $P_2(x+h, f(x+h))$



$$2 \text{ points}, m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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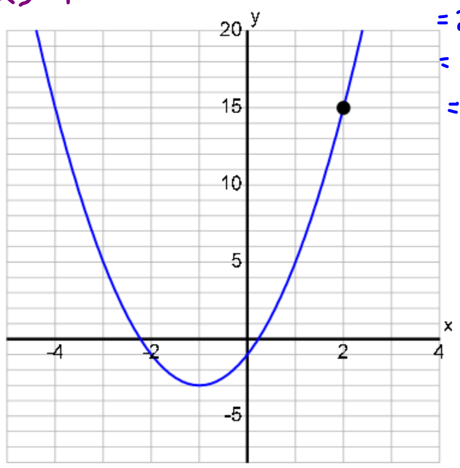
1. Given: $y = 2x^2 + 4x - 1$
   a) Use First Principles to find the slope of the tangent line to the curve $y = 2x^2 + 4x - 1$ at the point (2, 15).
   b) Find the equation of the line tangent to $y = 2x^2 + 4x - 1$ at the point (2,15).

$P(2, 15)$ $\quad P(2+h, \quad)$

$f(2) = 2(2)^2 + 4(2) - 1$

$f(2+h) = 2(2+h)^2 + 4(2+h) - 1$
$= 2(4 + 4h + h^2) + 8 + 4h - 1$
$= 8 + 8h + 2h^2 + 8 + 4h - 1$
$= 2h^2 + 12h + 15$



$$m = \lim_{h \to 0} \frac{(2h^2 + 12h + 15) - (15)}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 12h}{h} \ \ldots \ \frac{h(2h+12)}{h} \qquad \frac{2h^2}{h} + \frac{12h}{h}$$

$$m = \lim_{h \to 0} 2h + 12$$

$$= 2(0) + 12$$

$$m = 12$$

b) $m = 12$ $\quad P(2, 15)$ $\quad$ "linear equation"

$$\frac{12}{1} = \frac{y - 15}{x - 2}$$

$$12x - 24 = y - 15$$

$$12x - y = 9$$

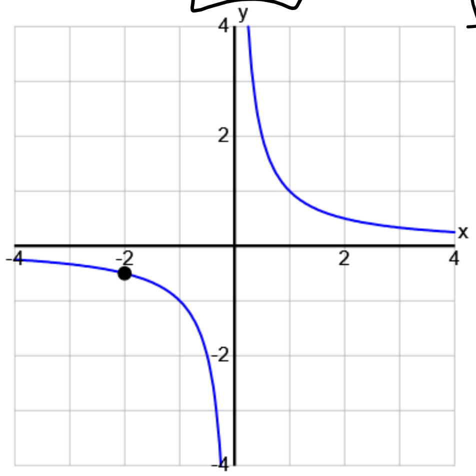
2. Find the tangent line to the hyperbola $\boxed{xy = 1}$ at the point (- 2 , - ½).

$f(-2+h) = \frac{1}{-2+h}$

$y = \frac{1}{x}$

$f(x) = \frac{1}{x}$

$f(-2) = -\frac{1}{2}$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$m = \lim_{h \to 0} \frac{\frac{1}{h-2} + \frac{1}{2}}{h} \left[\frac{2(h-2)}{2(h-2)}\right]$$

$$m = \lim_{h \to 0} \frac{2 + h - 2}{2h(h-2)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{2\cancel{h}(h-2)}$$

$$= \lim_{h \to 0} \frac{1}{2(h-2)}$$

$$= \frac{1}{2(0-2)} = \frac{1}{2(-2)}$$

$$m = -\frac{1}{4}$$

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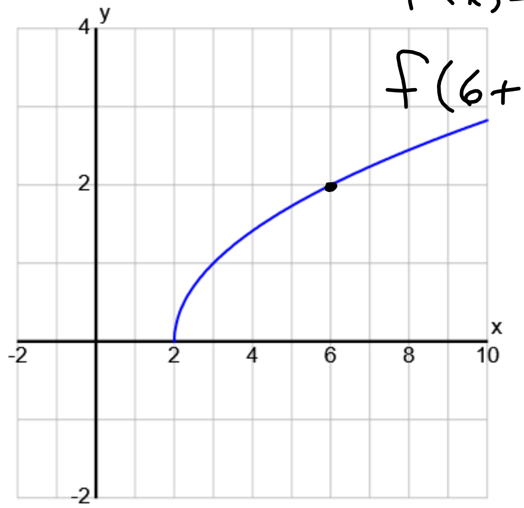
c) Find the slope of the tangent line to the curve $y = \sqrt{x-2}$ at the point where $x = 6$. Graph the tangent line.

$f(x) = \sqrt{x-2}$

$f(6) = \sqrt{6-2}$

$= \sqrt{4}$

$f(6) = 2$



$f(x) = \sqrt{x-2}$

$f(6+h) = \sqrt{6+h-2}$

$= \sqrt{h+4}$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \lim_{h \to 0} \frac{\sqrt{h+4} - 2}{h} \left(\frac{\sqrt{h+4} + 2}{\sqrt{h+4} + 2} \right)$$

$$m = \lim_{h \to 0} \frac{h+4-4}{h(\sqrt{h+4}+2)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{h+4}+2)}$$

$$m = \frac{1}{\sqrt{0+4}+2}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4}$$

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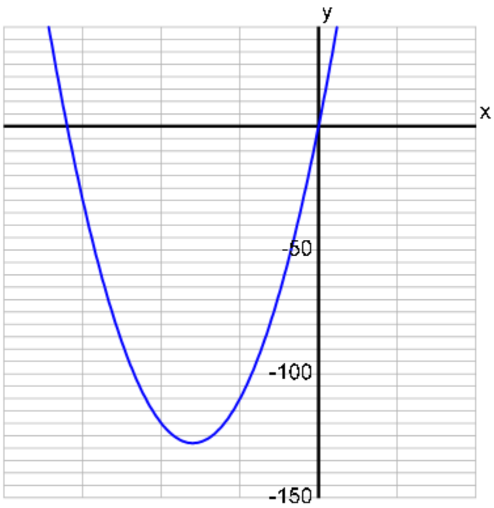
$f(x) = 2x^2 + 32x$

6. Find an equation to find the slope of a tangent line at any point tangent to the curve $y = 2x^2 + 32x$

$F(x+h) = 2(x+h)^2 + 32(x+h)$

$= 2(x^2 + 2xh + h^2) + 32x + 32h$

$= 2x^2 + 4xh + 2h^2 + 32x + 32h$



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \lim_{h \to 0} \frac{(2x^2 + 4xh + 2h^2 + 32x + 32h) - (2x^2 + 32x)}{h}$$

$$m = \lim_{h \to 0} \frac{4xh + 2h^2 + 32h}{h}$$

$$m = \lim_{h \to 0} 4x + 2h + 32$$

$$= 4x + 2(0) + 32$$

$$m = 4x + 32$$

$P(0,0)$ $m(0) = 4(0) + 32$

$m = 32$

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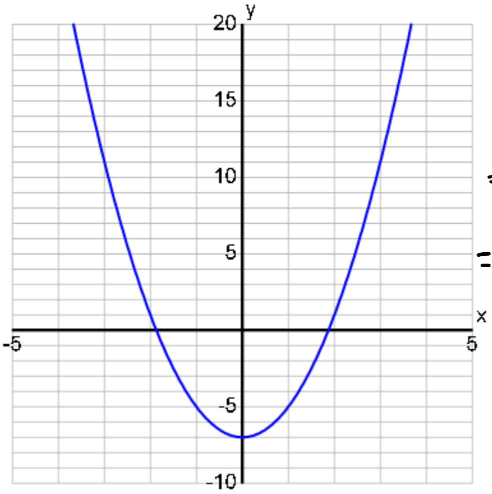
7. If $f(x) = 2x^2 - 7$, what is the value of x where the slope is 2?

$f(x) = 2x^2 - 7$



$f(x+h)$
$= 2(x+h)^2 - 7$
$= 2(x^2 + 2xh + h^2) - 7$
$= 2x^2 + 4xh + 2h^2 - 7$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$m = \lim\limits_{h \to 0} \dfrac{(2x^2 + 4xh + 2h^2 - 7) - (2x^2 - 7)}{h}$

$= \lim\limits_{h \to 0} \dfrac{4xh + 2h^2}{h}$

$= \lim\limits_{h \to 0} 4x + 2h$

$m = 4x + 2(0)$

$m = 4x$

$m = 2 \quad \ldots \quad m = 4x$

$2 = 4x$

$x = \frac{1}{2}$