The Limits of Trigonometric Expressions

Review Limits (evaluate):

$\lim_{x\to 5}\sqrt{x+4}$ $= \sqrt{5+4}$

$= 3$

not ÷ zero

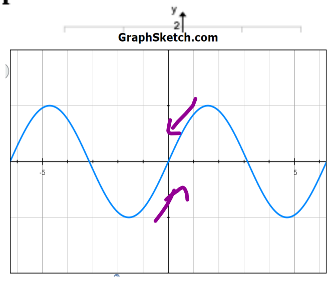
$\lim_{x\to 5}\frac{x^2-6x+5}{x^2-25} = \lim_{x\to 5}\frac{\cancel{(x-5)}(x-1)}{\cancel{(x-5)}(x+5)}$

÷ zero

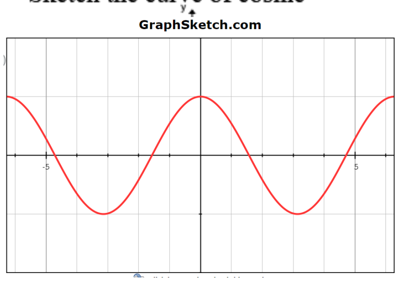
$= \frac{5-1}{5+5} = \frac{4}{10} = \frac{2}{5}$

Outcomes: Find the limits of sine and cosine and simple modifications to them.

Warm up: Sketch the curve of sine



Sketch the curve of cosine



Investigate: Look at the left- and right-hand limits of $\sin\theta$ and $\cos\theta$ as $x\to 0$

Since we know that sine and cosine are continuous graphs state the value of

$\lim_{\theta\to 0}\sin\theta = 0$ and $\lim_{\theta\to 0}\cos\theta$. $= 1$

Examples:

ALL CONTINUOUS ∴

$\lim_{x\to a} = f(a)$

1. Evaluate limits, not divide by zero.

a) $\lim_{\theta\to\pi}\frac{\sin\theta}{2} = \frac{\sin\pi}{2}$

$= \frac{0}{2}$

$= 0$

b) $\lim_{x\to 0}(\sin x + x)$.

$= \sin(0) + 0$

$= 0 + 0$

$= 0$

c) $\lim_{x\to\pi}(\sin x + \cos x)$

$= \sin\pi + \cos\pi$

$= 0 + (-1)$

$= -1$

d) $\lim_{x\to\frac{\pi}{2}}\frac{\sin x + 1}{\cos x + 1}$

$= \frac{\sin\frac{\pi}{2} + 1}{\cos\frac{\pi}{2} + 1}$

$= \frac{1+1}{0+1}$

$= 2$

e) $\lim_{x\to\frac{3\pi}{2}}\frac{\cos x}{2x}$

$= \frac{\cos\frac{3\pi}{2}}{2\left(\frac{3\pi}{2}\right)}$

$= \frac{0}{3\pi}$

$= 0$

f) $\lim_{x\to 0}\frac{\cos 2x}{3\cos 3x}$

$= \frac{\cos(2\cdot 0)}{3\cos(3\cdot 0)}$

$= \frac{1}{3(1)}$

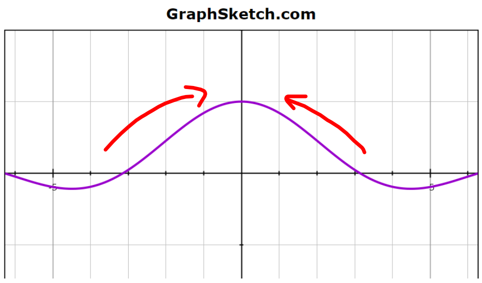
$= \frac{1}{3}$

Investigate: Some limits which will be very important to trigonometric functions are:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \quad \text{or} \quad \lim_{\theta \to 0} \frac{\theta}{\sin\theta} \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

These limits are necessary in order to find the derivatives of trigonometric functions.

Graph: $y = \dfrac{\sin\theta}{\theta}$ What does the graph indicate the $\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta}$ is equal to?

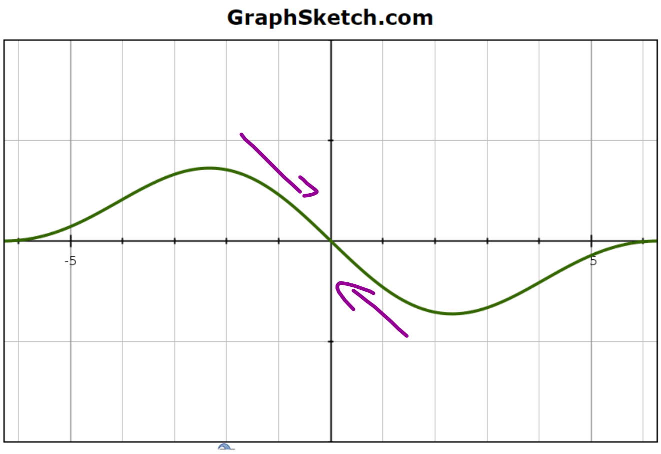


$\lim_{\theta \to 0^-} f(\theta) = 1$

$\lim_{\theta \to 0^+} f(\theta) = 1$

$\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Graph: $y = \dfrac{\cos\theta - 1}{\theta}$ What does the graph indicate the $\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta}$ is equal to?



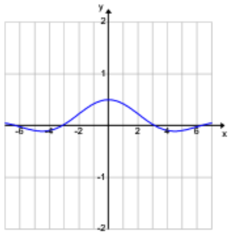
$\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

$\frac{0}{0} = \frac{\sin\theta}{\theta} = 1$ $\frac{0}{0} = \frac{\cos\theta - 1}{\theta} = 0$

Examples:

2. Evaluate limits, dividing by zero: Graphically to see Algebraically to justify when $\left(\frac{0}{0}\right)$.

a) $\lim\limits_{\theta\to 0}\frac{\sin\theta}{2\theta} = \frac{1}{2}$

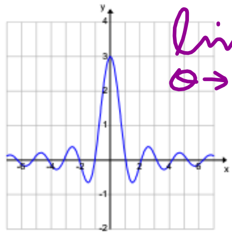


$\lim\limits_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\right]\frac{1}{2}$

$= (1)\left(\frac{1}{2}\right)$

$= \frac{1}{2}$

b) $\lim\limits_{\theta\to 0}\frac{\sin 3\theta}{\theta} = 3$



$\lim\limits_{\theta\to 0}\left[\frac{\sin(3\theta)}{3\theta}\right]\frac{3}{1}$

$= [1][3]$

$= 3$

c) $\lim\limits_{\theta\to 0}\frac{\sin^2\theta}{\theta}$

$= \lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin\theta}{1}\right)$

$= [1][\sin 0]$

$= [1][0]$

$= 0$

d) $\lim\limits_{x\to 0} x\sec x$. not $\frac{0}{0}$

$= (0)(\sec 0)$

$= (0)(1)$

$= 0$

$\frac{0}{0}: \frac{\sin\theta}{\theta} = 1$

e) $\lim\limits_{x\to 0}\frac{\tan x}{\sin x}$.

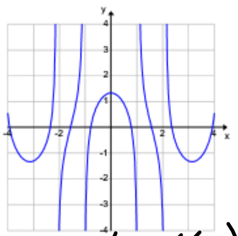
$= \lim\limits_{x\to 0}\frac{\frac{\sin x}{\cos x}}{\sin x}$ ... $\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right)$

$= \lim\limits_{x\to 0}\frac{1}{\cos x}$

$= \frac{1}{1}$

$= 1$

f) $\lim\limits_{x\to 0}\frac{\sin 4x}{\sin 3x} = \frac{4}{3}$



$$\frac{\left[\frac{\sin 4x}{4x}\right]\frac{4x}{1}}{\left[\frac{\sin 3x}{3x}\right]\left[\frac{3x}{1}\right]}$$

$$\lim_{x\to 0}\frac{\left(\frac{\sin 4x}{4x}\right)(4)}{\left(\frac{\sin 3x}{3x}\right)(3)}$$

$$= \frac{(1)(4)}{(1)(3)}$$

$$= \frac{4}{3}$$

g) $\lim_{\theta \to \frac{\pi}{2}} \frac{\sin\theta}{2\theta}$ not $\frac{0}{0}$

$$= \frac{\sin \frac{\pi}{2}}{2\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{\pi}$$

h) $\lim_{x \to 0} \frac{\sin^2 3x}{\sin^2 4x}$

$$= \lim_{x \to 0} \left[\frac{\left(\frac{\sin 3x}{3x}\right)\left(\frac{3x}{1}\right)}{\left(\frac{\sin 4x}{4x}\right)\left(\frac{4x}{1}\right)} \right]^2$$

$$= \left[\frac{(1)(3)}{(1)(4)} \right]^2$$

$$= \frac{9}{16}$$

$\frac{0}{0}$ i) $\lim_{x \to \frac{\pi}{4}} \frac{\cos 2x}{\cos x - \sin x}$

$$= \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cancel{(\cos x - \sin x)}(\cos x + \sin x)}{\cancel{(\cos x - \sin x)}}$$

$$= \cos \frac{\pi}{4} + \sin \frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= \sqrt{2}$$

j) $\lim_{x \to 0} \frac{\tan x}{\tan 2x}$

$$= \lim_{x \to 0} \frac{\tan x}{\frac{2\tan x}{1 - \tan^2 x}}$$

$$\left(\frac{\tan x}{1}\right)\left(\frac{1 - \tan^2 x}{2\tan x}\right)$$

$$= \lim_{x \to 0} \frac{1 - \tan^2 x}{2}$$

$$= \frac{1 - 0}{2} = \frac{1}{2}$$

Homework: Page 306: 1,2,7,9, 11,12,13,15,16,17,18,19,20,21,23,27,31,33