4.0 Limits for continuity

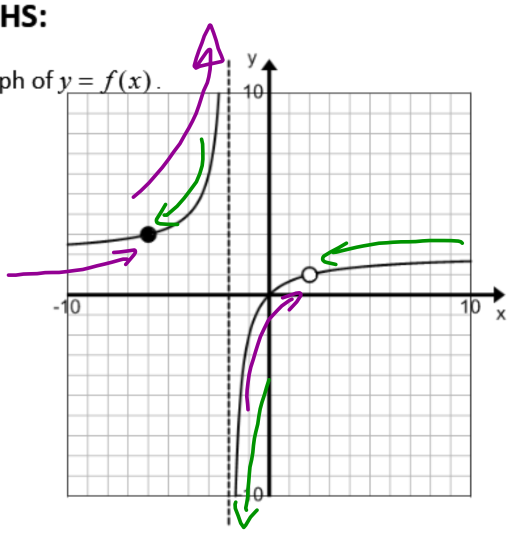
One Sided Limits & Discontinuity

Outcomes:

- Find limits using the graph of a function.
- Determine if a function is continuous or not for specific values of *x*.
- Evaluate expressions using one-sided limits to determine if a limit exists:
    - left-hand $\left(\lim_{x \to a^-}\right)$ limit as x approaches from the left side of 'a'
    - right-hand $\left(\lim_{x \to a^+}\right)$ limit as x approaches from the right side of 'a'
- Show that a function is continuous at point *'a'* using the property $\lim_{x \to a} f(x) = f(a)$

USING GRAPHS:

1. Given the graph of $y = f(x)$.



For the closed point, the vertical asymptote and the open point:

- Find one sided limits for each x value
- Determine if a limit exits at each x value
- Determine if the function is continuous or not continuous at each value of x.

"Limit Exists: $\lim_{x \to a} f(x)$. Point Exists: $P(x, f(x))$. Limit = Point: $\lim_{x \to a} f(x) = f(x)$."

$\lim_{x \to -6^-} f(x) = 3$

$\lim_{x \to -6^+} f(x) = 3$

$\lim_{x \to -6} f(x) = 3$

$f(-6) = 3$

$\lim_{x \to -2^-} f(x) = +\infty$

$\lim_{x \to -2^+} f(x) = -\infty$

$\lim_{x \to -2} f(x) =$ Undefined
does not exist

$\lim_{x \to 2^-} f(x) = 1$

$\lim_{x \to 2^+} f(x) = 1$

$\lim_{x \to 2} f(x) = 1$

$f(2) = u$

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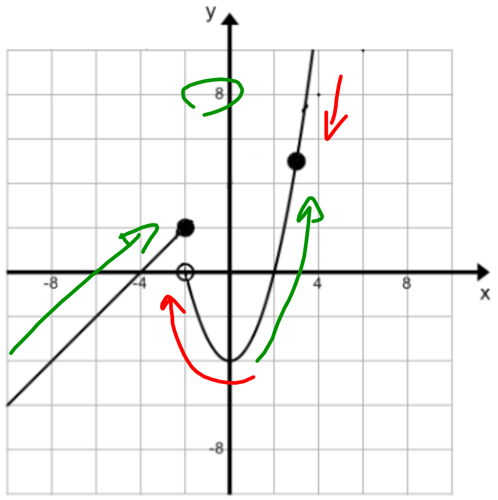
2. Given the graph of $y = f(x)$.

$$f(x) = \begin{cases} x+4 & \text{if } x \le -2 \\ x^2 - 4 & \text{if } x > -2 \end{cases}$$



Use one sided notation to determine if the following limits exist:

$\lim_{x \to -2^-} f(x) = 2 \ldots (-2)+4$

$\lim_{x \to -2^+} f(x) = 0 \ldots (-2)^2 - 4$

$\lim_{x \to 3^-} f(x) = 5 \ldots (3)^2 - 4$

$\lim_{x \to 3^+} f(x) = 5 \ldots (3)^2 - 4$

$\lim_{x \to -2} f(x) =$ Undefined

Point $= f(-2) = -2+4 = 2$

$\lim_{x \to 3} f(x) = 5$

$f(3) = 5$

Use properties of continuous functions to determine if the function is continuous at: $f(-2)$; $f(3)$?

continuous at $x=3$

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Combine graphing and algebraic reasoning to determine if limits exist, to determine if a function is continuous. Note: Functions are not always created by single equations.

1. $f(x)=\begin{cases} x^2 & if\ x\le 1 \\ 3-x & if\ x>1 \end{cases}$

   a) Find the values for the function from both 'parts'

$x \le 1$

domain

$x > 1$

$f(-2) = (-2)^2 = 4$

$f(-1) = (-1)^2 = 1$

$f(0) = (0)^2 = 0$

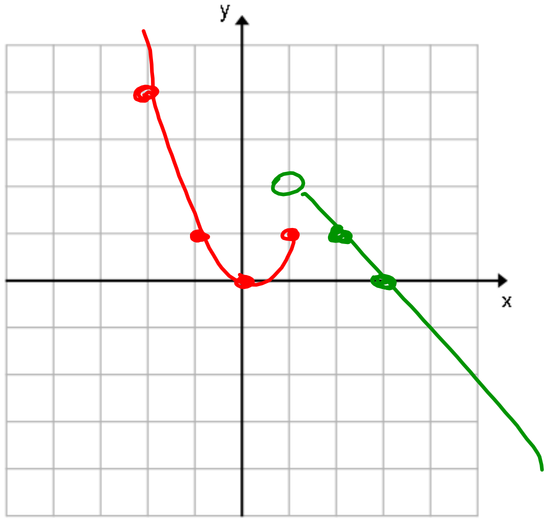
$f(1)$... closed circle. $= (1)^2 = 1$

$f(3) = 3-3 = 0$

$f(2) = 3-2 = 1$

$f(1.01) = 3-1.01 = 1.99$

$f(1)$ ... open circle. $= 3-1 = 2$



   b) Sketch the function [key x's are where we have open and/or closed circle points and/or vertical asymptotes].

Find limits and determine if the function is continuous for x.

   c) Find $\lim_{x\to 2^-} f(x) = 3-2 = 1$ $\quad \lim_{x\to 2^+} f(x) = 3-2 = 1$ $\quad \boxed{\lim_{x\to 2} f(x) = 1 \quad f(2) = 1}$ continuous

   d) Find $\lim_{x\to 1^-} f(x) = (1)^2 = 1$ $\quad \lim_{x\to 1^+} f(x) = 3-1 = 2$ $\quad \boxed{\lim_{x\to 1} f(x) = \text{Undefined}}$

$f(1) = 1$

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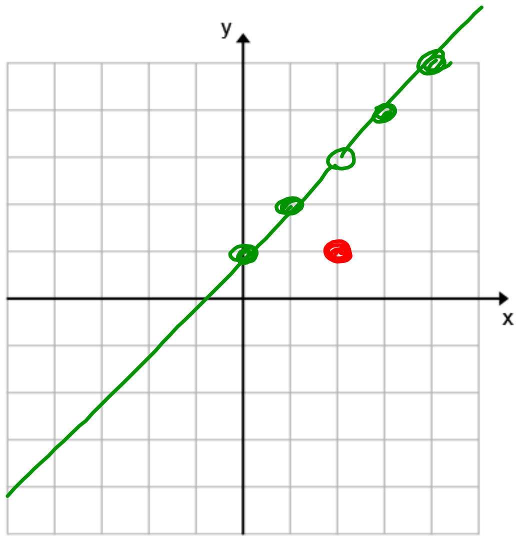
$y = x + 1$

2. Given: $f(x) = \begin{bmatrix} x+1 & x \neq 2 \\ \boxed{1 \quad x = 2} \end{bmatrix}$

$f(2) = 1$

a) Graph $y = f(x)$



b) Find the limits:

$\lim_{x \to 2^-} f(x) = x + 1 = 2 + 1 = 3$

$\lim_{x \to 2^+} f(x) = x + 1 = 2 + 1 = 3$

$\lim_{x \to 2} f(x) = 3$ $\qquad f(2) = 1$

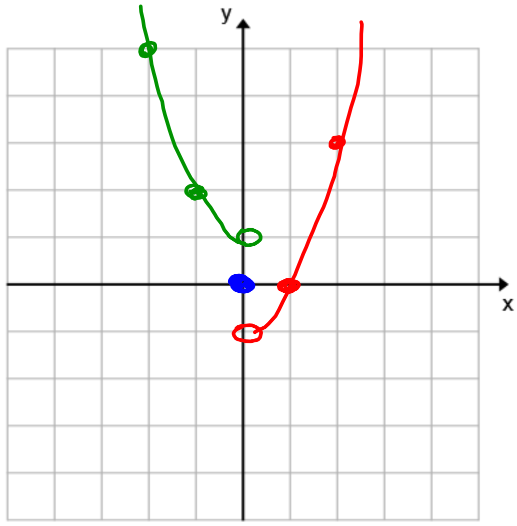
c) Is the function continuous where $x = 2$?

Not continuous $\quad \lim_{x \to 2} f(x) = f(2)$

3. Given: $f(x) = \begin{bmatrix} x^2 + 1 & \boxed{x < 0} \\ \boxed{0 \quad x = 0} \\ x^2 - 1 & \boxed{x > 0} \end{bmatrix}$

open (0,1)
$f(0) = 0$
open (0,-1)

a) Graph $y = f(x)$



b) Find the limits:

$\lim_{x \to 0^-} f(x) = (0)^2 + 1 = 1$

$\lim_{x \to 0^+} f(x) = (0)^2 - 1 = -1$

$\lim_{x \to 0} f(x) =$ undefined

Point (0,0)

c) Is the function continuous where $x = 0$?

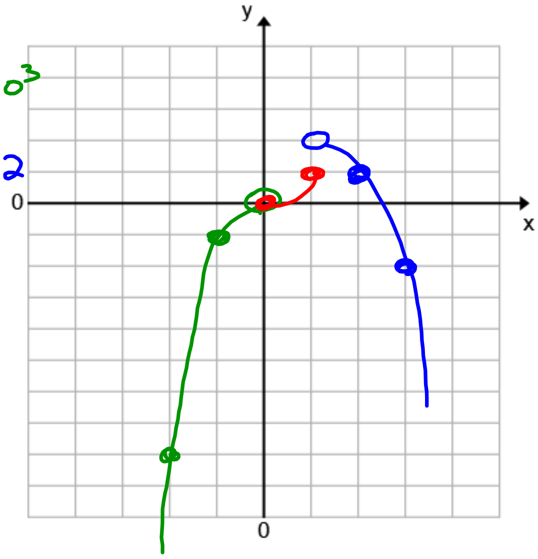
limit $\neq$ point

4.0 Limits for continuity

4. Graph $f(x) = \begin{bmatrix} x^3 & x<0 \\ x^2 & 0 \le x \le 1 \\ 1+2x-x^2 & \boxed{x>1} \end{bmatrix}$

open
$f(0) = 0^3$
open
$f(1):2$

$f(0) = (0)^2 = 0 \quad f(1) = (1)^2 = 1$



a) Graph $y = f(x)$ [key point(s) at boundary]

b) Find the limits:

$\lim_{x\to 0^-} f(x) = x^3 = (0)^3 = 0$

$\lim_{x\to 0^+} f(x) = x^2 = (0)^2 = 0$

$\lim_{x\to 0} f(x) = 0$

$f(0) = 0$

c) Is the function $\boxed{\text{continuous}}$ where $x = 0$?

yes: $\lim_{x\to 0} f(x) = f(0)$

d) Find the limits:

$\lim_{x\to 1^-} f(x) = x^2 = (1)^2 = 1$

$\lim_{x\to 1^+} f(x) = 1 + 2x - x^2 = 1 + 2(1) - (1)^2 = 2$

$\lim_{x\to 1} f(x) = $ undefined

$f(1) = 1$

e) Is the function continuous where $x = 1$?

No: limit $\neq$ point

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5. Show that $\lim_{x\to 0}|x| = 0$

$$f(x) = \begin{bmatrix} -x & x<0 \\ x & x\geq 0 \end{bmatrix}$$

$$\lim_{x\to 0^-} f(x) = -(0) = 0$$

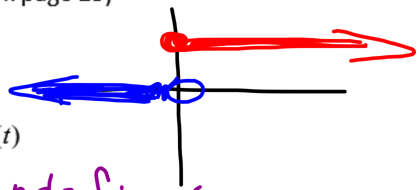
$$\lim_{x\to 0^+} f(x) = (0) = 0$$

6. The **Heaviside function** H is defined by (details are in your book page 23)

$$H(t)=\begin{Bmatrix} 0 & t<0 \\ 1 & t\geq 0 \end{Bmatrix}$$

$y=0$

$y=1$ P(0,1)



Find $\lim_{x\to 0^-} H(t) = 0$ $\quad \lim_{x\to 0^+} H(t) = 1$ $\quad \lim_{x\to 0} H(t)$

= undefined

equal "point"

7. If $f(x)=\begin{Bmatrix} -x-2 & x\leq -1 \\ x & -1<x<1 \\ x^2-2x & x\geq 1 \end{Bmatrix}$ determine whether or not $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ exist.

$$\lim_{x\to -1^-} f(x) = -x-2 = -(-1)-2 = -1$$

$$\lim_{x\to -1^+} f(x) = x = (-1) = -1$$

$$f(-1) = -x-2$$
$$= -(-1)-2$$
$$f(-1) = -1$$

Homework: Page 27 # 1,2,4,5,6,7

Continuous at $x=-1$

$$\lim_{x\to -1} f(x) = f(-1)$$