4.0 Rates.2019

Rates of Change

1. A spherical balloon is being inflated. The volume is changing with respect to the balloon's radius. Find the rate of change of the balloon's volume when its radius is 15cm.

2. The population of a bacteria culture after t hours is given by $n(t)=500+200t-12t^2$ where t is time in hours.
   a) Find the rate of growth after 5 hours.
   b) Is the population increasing or decreasing?
   c) What is the maximum population of this culture?

Rate: $\frac{\text{pop}}{\text{time}} = \frac{\Delta n}{\Delta t}$

$\therefore \frac{d}{dt}$

Homework: Page 133 # 1,2,3,6

sphere $V=\frac{4}{3}\pi r^3$ $\quad SA=4\pi r^2$

cylinder $V=\pi r^2 h$ $\quad SA=2\pi r^2+2\pi rh$

cube $V=l^3$ $\quad SA=6l^2$

1. $\frac{\Delta V}{\Delta r} \Rightarrow \frac{dV}{dr} = \frac{d}{dr}(V)$

$\frac{d}{dr}\left[V=\frac{4}{3}\pi r^3\right]$

$\frac{dV}{dr}=\frac{4}{3}\pi\left[3r^2\right]$

$\frac{dV}{dr}=4\pi r^2$

★ we need derivatives before solving for "instance" in time.

$r=15$

$\frac{dV}{dr}=4\pi(15)^2$

$\frac{dV}{dr}=900\pi \ \frac{cm^3}{cm}$

2. The population of a bacteria culture after t hours is given by $n(t)=500+200t-12t^2$ where t is time in hours.
   a) Find the rate of growth after 5 hours.
   b) Is the population increasing or decreasing?
   c) What is the maximum population of this culture?

$\frac{d}{dt}\left[n=500+200t-12t^2\right]$

$\frac{dn}{dt}=n'(t)=200-24t$

$n'(5)=200-24(5)$ $\quad 120$

$n'(5)=80$ bacteria/h

b) positive rate $\therefore$ growing population.

c) MAX when $\frac{dn}{dt}=0$

$0=200-24t$

$t=\frac{25}{3}$

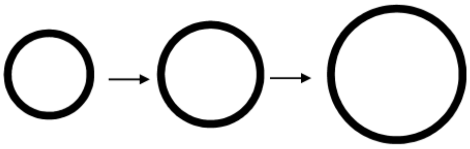
$n\left(\frac{25}{3}\right)=500+200\left(\frac{25}{3}\right)-12\left(\frac{25}{3}\right)^2$ $\quad 12\left(\frac{25}{3}\right)\left(\frac{25}{3}\right)$ $\quad \frac{4(25)(25)}{3}$

$=\frac{1500}{3}+\frac{5000}{3}-\frac{2500}{3}$

$=\frac{4000}{3}$

$=1333$ bacteria

3. A circular ring is heated so that it expands. If the rate of increase of the radius is $\boxed{0.01\text{ cm/s}}$, determine the rate at which the circumference is increasing. $\frac{d}{dt}$



4. This same disk is now cooling. During the cooling process, the radius is found to be decreasing at a rate of 0.02 cm/s. At what rate is the area of the disk changing when the radius of the disk is 100 cm.

3. $\frac{d}{dt}\left[C = 2\pi r\right]$

2 VAR = 2 rates

$\frac{dV}{dt}, \frac{dr}{dt}$

$$\frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$$

$$\frac{dC}{dt} = 2\pi(0.01\text{ cm/s})$$

$$= 0.02\pi\text{ cm/s}$$

4. $\frac{d}{dt}\left[A = \pi r^2\right]$ … $\frac{dA}{dt}, \frac{dr}{dt}$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

* need to know "r"

$r = 100\text{ cm}$

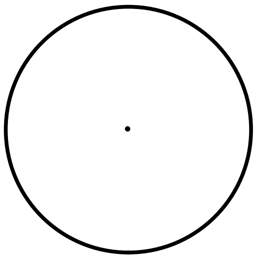
$\frac{dr}{dt} = -0.02\text{ cm/s}$

$$\frac{dA}{dt} = 2\pi(100\text{ cm})(-0.02\text{ cm/s})$$

$$\frac{dA}{dt} = -4\pi\text{ cm}^2\text{/s}$$

5. If the radius of a circle is increasing at the rate of 7 cm/s, how fast is its area increasing when the radius is 20 cm?

RATE $\frac{d}{dt}$



$A = \pi r^2$

2 VAR : 2 RATES

$\frac{dA}{dt}$ $\frac{dr}{dt}$

$\frac{d}{dt}\left[A = \pi r^2\right]$

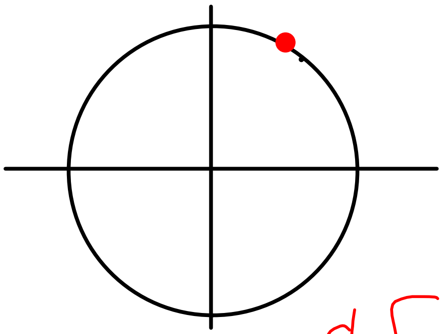
$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

given: $r = 20$

$\frac{dr}{dt} = 7 \text{ cm/s}$

$\frac{dA}{dt} = 2\pi(20)(7) = 280\pi \text{ cm}^2/\text{s}$

6. An object is moving clockwise around the circle $x^2 + y^2 = 100$. As it passes through the point (6,8) the y- coordinate is decreasing at the rate of 3 units per second. At what rate is its x – coordinate changing at that point.



2 VAR — 2 RATES

x $\frac{dx}{dt}$

y $\frac{dy}{dt}$

$\frac{d}{dt}\left[x^2 + y^2 = 100\right]$

$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$

$P(x,y) = (6,8)$

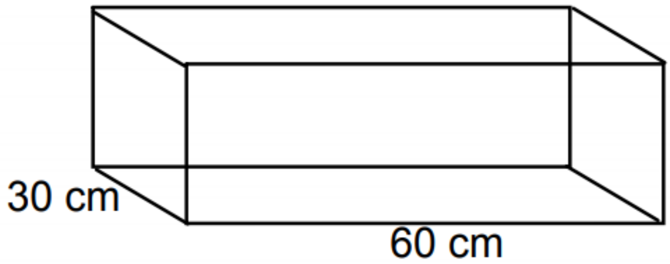
$\frac{dy}{dt} = -3$

$(6)\frac{dx}{dt} + 8(-3) = 0$

$6\frac{dx}{dt} = 24$

$\frac{dx}{dt} = 4$

7. Water is being poured into an aquarium that is 60 cm long, 30cm wide and 40 cm deep. Determine a relationship between the rate at which water is being poured and the rate at which the depth is increasing. (NOTE: You can substitute the appropriate constants before you differentiate)



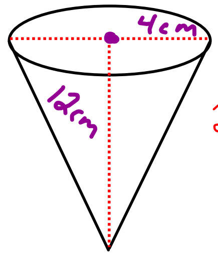
$V = \underbrace{lw}_{\text{constant}}h$

$V = (60)(30)h$

$\frac{d}{dt}\left[V = 1800h\right]$

$\frac{dV}{dt} = 1800\frac{dh}{dt}$

8. A conical paper cup 8 cm across the top and 12cm deep is full of water. Water begins to leak out of the bottom at the rate of 2 cm^3/min. How fast is the level of water dropping when the water is 3 cm deep.



$V = \frac{1}{3}\pi r^2 h$

2 RATES

$\frac{cm^3}{min} = \frac{dV}{dt}$

dropping $= \frac{dh}{dt}$

3 VAR

V

$\boxed{r}$

h

ΔS

$\frac{r}{h} = \frac{4}{12}$

$\frac{r}{h} = \frac{1}{3}$ $\therefore$ $\boxed{r = \frac{1}{3}h}$

$V = \frac{1}{3}\pi r^2 h \Longrightarrow V(h) = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{9}h^2\right)(h)$

I. VAR = RATES

II. RATES

$V = \frac{1}{27}\pi h^3$

$\frac{d}{dt}\left(V = \frac{1}{27}\pi h^3\right)$

$\frac{dV}{dt} = \frac{1}{27}\pi(3h^2)\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

III. solve "moment in time"

$h = 3$ cm

$\frac{dV}{dt} = -2\ cm^3/min$ (water out)

$-2 = \frac{1}{9}\pi(3)^2 \frac{dh}{dt}$

$-2 = \cancel{\frac{1}{9}}\pi(\cancel{9})\frac{dh}{dt}$

$-\frac{2}{\pi} = \frac{dh}{dt}$

"hundredths"

$\frac{dh}{dt} = -0.63\ cm/min$

$\frac{dA}{dt}$

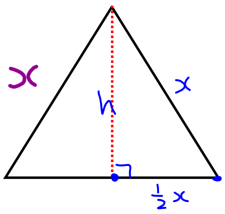
9. The area of an equilateral triangle is increasing at a rate of 14 cm²/min. Find the rate the length of each side is increasing when the area is $400\sqrt{3}\ cm^2$

Homework: Page 145 # 1 – 9

$A = \frac{bh}{2}$

2 RATES

$\frac{dA}{dt}$ $\frac{dx}{dt}$



base $= x$

$(\frac{1}{2}x)^2 + (h)^2 = (x)^2$

$\frac{1}{4}x^2 + h^2 = x^2$

$h^2 = \frac{3}{4}x^2$

$\frac{\sqrt{3}}{\sqrt{4}}\ \sqrt{x^2}$

height $= \frac{\sqrt{3}}{2}x$

$$A = \frac{(x)\left(\frac{\sqrt{3}}{2}x\right)}{2}$$

$$A = \frac{\sqrt{3}}{4}x^2$$

I. VAR = RATES

II. Rates

$$\frac{d}{dt}\left[A = \frac{\sqrt{3}}{4}x^2\right]$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}(2x)\frac{dx}{dt}$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}x \cdot \frac{dx}{dt}$$

$\boxed{A = 400\sqrt{3}\ cm^2}$

III. moment in time

$\frac{dA}{dt} = 14\ cm^2/min$

FIND x on own...

$A = \frac{\sqrt{3}}{4}x^2$

$400\sqrt{3} = \frac{\sqrt{3}}{4}x^2$

$1600 = x^2$

$x = 40$

$$14 = \frac{\sqrt{3}}{2}(40)\frac{dx}{dt}$$

$$14 = 20\sqrt{3}\ \frac{dx}{dt}$$

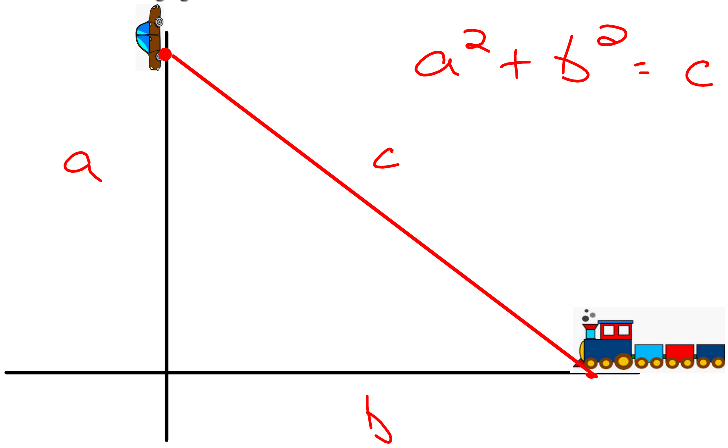
$$\frac{dx}{dt} = \frac{14}{20\sqrt{3}}$$

$$\frac{dx}{dt} = \frac{7}{10\sqrt{3}}$$

...hundredths

$$\frac{dx}{dt} = 0.41\ cm/min$$

10. A straight level road crosses a railroad track at a right angle. A car is on the road 1 km from the crossing, traveling at 80km/h toward the crossing. At the same time, a train, 1.2 km from the crossing, is traveling 95 km/h toward the crossing. At what rate is the distance between the car and the train changing at that instant?



$a^2+b^2=c^2$

I VAR = RATES ✓

II. Rates

$$\frac{d}{dt}\left[a^2+b^2=c^2\right]$$

$$\cancel{2}a\cdot\frac{da}{dt}+\cancel{2}b\cdot\frac{db}{dt}=\cancel{2}c\cdot\frac{dc}{dt}$$

III. moment in time:

$a=1.0\text{ km}$ $\qquad \frac{da}{dt}=80\text{ km/h}$

$b=1.2\text{ km}$ $\qquad \frac{db}{dt}=95\text{ km/h}$

FIND "c" on own. $\qquad \frac{dc}{dt}=?$

$1.0^2+1.2^2=c^2$

$c^2=2.44$

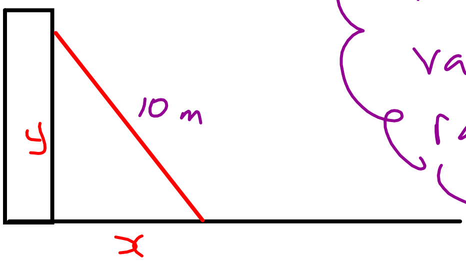
$c=\sqrt{2.44}$

$$(1.0)(80)+(1.2)(95)=(\sqrt{2.44})\frac{dc}{dt}$$

$$\frac{dc}{dt}=124\text{ km/h}$$

Distance is decreasing at 124 km/h.

11. A 10m ladder is leaning against a wall. It begins to slide down the wall. How fast is the bottom of the ladder sliding away from the base of the wall when the top of the ladder reaches 6m up the wall and is falling at a rate of 2m/s?



formula
variables
rates
...

$a^2+b^2=c^2$

$$x^2+y^2=100$$

VAR = RATES ✓

$$\frac{d}{dt}\left[x^2+y^2=100\right]$$

RATES

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

III. $y=6$ $\quad \frac{dy}{dt}=-2\,m/s$

$$x^2+6^2=100$$

$$x=8$$

$$(8)\frac{dx}{dt}+(6)(-2)=0$$

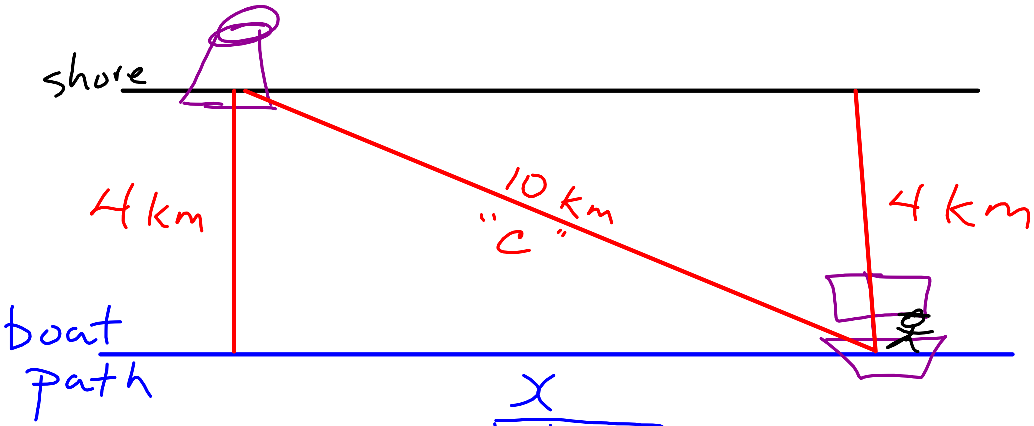
$$\frac{dx}{dt}=\frac{12}{8}=\frac{3}{2}$$

$$\frac{dx}{dt}=1.5\,m/s$$

Ladder slides away at 1.5 m/s.

12. Two cars leave the city at the same time form the same location. One travels east at 100km/h and the other travels south at 110km/h. How fast are they separating after 1.5 h?

13. A boat is cruising parallel to the shore and is 4km from the shoreline. A radar tracking device, situated in a lighthouse on the shore, calculates the distance between them to be decreasing at a rate of 15 km/h, when they are 10 km apart. What is the velocity of the boat?



$$\boxed{\frac{dx}{dt}}$$

$$a^2+b^2=c^2$$

$$\frac{d}{dt}\left[4^2+x^2=c^2\right]$$

VAR = RATES

RATES

$$2x\frac{dx}{dt}=2c\frac{dc}{dt}$$

Moment: $c=10$ $\quad \frac{dc}{dt}=-15$ km/h

on my own "x"

$$4^2+x^2=10^2$$

$$x^2=84$$

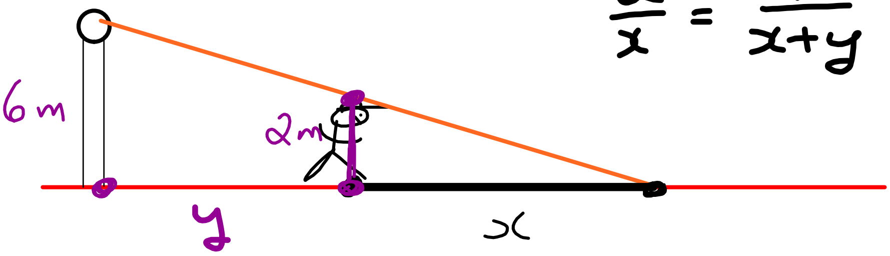
$$\sqrt{84}\,\frac{dx}{dt}=(10)(-15)$$

negative has no context...

$$\frac{dx}{dt}=16.4$$

Boat is moving towards lighthouse at 16.4 km/h

14. A jogger, 2m tall, passes a streetlight 6m high, at a rate of 1.25 m/s. At what rate is the shadow of the jogger increasing in length? At what rate is the tip of the shadow moving?



$$\frac{2}{x} = \frac{6}{x+y}$$

$\frac{dy}{dt} = 1.25\,m/s$ $\quad \frac{dx}{dt}$

"similar triangles"

$$\frac{\text{small }\Delta}{\text{big }\Delta} : \frac{2}{6} = \frac{x}{x+y}$$

$$2x + 2y = 6x$$

$$2y = 4x$$

$$\left[\, y = 2x \,\right] \frac{d}{dt}$$

$$\frac{dy}{dt} = 2\frac{dx}{dt}$$

$$1.25 = 2\frac{dx}{dt}$$

$$\frac{dx}{dt} = 0.625\ m/s$$ shadow increasing speed

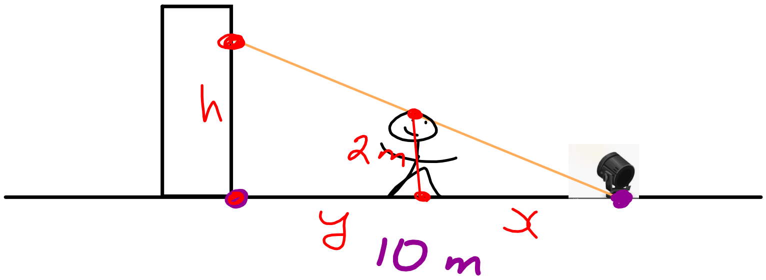
TIP SPEED $= 1.25 + 0.625$

$= 1.875$

$= 1.9\ m/s$

15. A spotlight on the ground shines on a wall 10 m away. A man 2 m tall walks from the spotlight toward the wall at a speed of 1.2m/s. How fast is his shadow on the wall decreasing when he is 3 m from the wall?

Homework: Page 145 # 10 - 16



similar △

$$\frac{2}{x} = \frac{h}{x+y}$$

2 RATES
3 VAR

$\frac{dh}{dt}$ $\cancel{\frac{dy}{dt}}$ $\frac{dx}{dt}$

$x+y=10$

$$\frac{2}{x} = \frac{h}{10}$$

$$\frac{d}{dt}\left[20 = xh\right]$$ product rule...

$$0 = (1)\left(\frac{dx}{dt}\right)(h) + (x)(1)\frac{dh}{dt}$$

3m from wall $\therefore$ $x+y=10$

$y=3$ $x=7$

$\frac{dx}{dt} = 1.2\,m/s$ $h = ?$

$20 = xh$

$20 = 7h$

$h = \frac{20}{7}$

$$(1.2)\left(\frac{20}{7}\right) + (7)\frac{dh}{dt} = 0$$

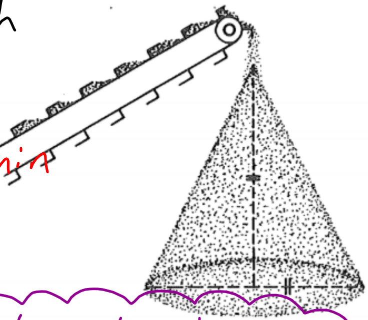
$$\frac{24}{7} + 7\frac{dh}{dt} = 0$$

$$7\frac{dh}{dt} = -\frac{24}{7}$$

$$\frac{dh}{dt} = -\frac{24}{49} \quad \text{OR} \quad -0.49\,m/s$$

17. Sand is being dumped from a conveyor belt at a rate of 1.2 m³/min and forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile growing when the pile is 3 m high?

$V = \frac{1}{3}\pi r^2 h$



$\frac{dV}{dt} = 1.2\ m^3/min$

$\frac{dh}{dt} = ?$

diameter = height

$2r = h$

$r = \frac{1}{2}h$

$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)(h)$

$V = \frac{1}{12}\pi h^3$

I RATE = VAR

II Rates

III moment

$h = 3m$

$\frac{dV}{dt} = 1.2\ m^3/min$

$\frac{d}{dt}\left[V = \frac{1}{12}\pi h^3\right]$

$\frac{dV}{dt} = \frac{1}{12}\pi(3h^2)\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$1.2 = \frac{1}{4}\pi(3)^2 \frac{dh}{dt}$

$1.2 = \frac{9\pi}{4}\frac{dh}{dt}$

$(1.2)\left(\frac{4}{9\pi}\right) = \frac{dh}{dt}$

$\frac{dh}{dt} = 0.17\ m/min$

$\boxed{OR}$

$17\ cm/min$