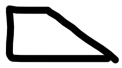
4. Area

Definite Integrals & Area Between Curves

- Use integrals to determine area below a curve (between function and the x-axis).
- Use integrals to determine area between curves.

Warm up: **Given the curve** $f(x) = 4x + 3$

Find the area, $A = \left(\frac{b_1 + b_2}{2}\right)(h)$, of the trapezoid formed between $f(x)$ and the x-axis, from



a) $x = 0$ *to* $x = 1$

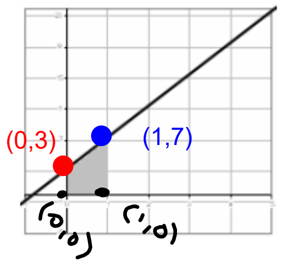
$A = \left(\frac{3+7}{2}\right)(1)$

$A = 5$

$b_1 = 3$

$b_2 = 7$

$h = 1$



b) $x = 0$ *to* $x = 5$

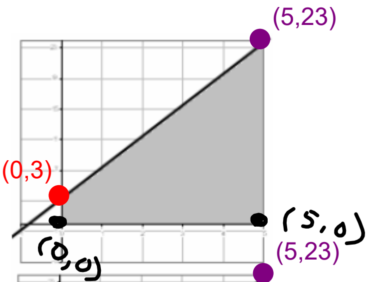
$A = \left(\frac{3+23}{2}\right)(5)$

$A = 65$

$b_1 = 3$

$b_2 = 23$

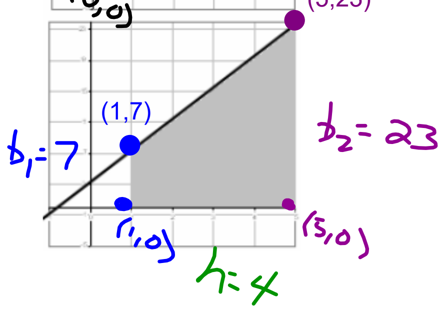
$h = 5$



c) $x = 1$ *to* $x = 5$

$A = \left(\frac{7+23}{2}\right)(4)$

$A = 60$



OR

$A = 65 - 5 = 60$

4. Area

Investigate:

Find the function $F(x)=\int(4x+3)dx$ and evaluate $F(5)-F(1)$.

$F(x)=\frac{4x^2}{2}+\frac{3x^1}{1}+c$

$F(x)=2x^2+3x+c$

$F(5)=2(5)^2+3(5)+c=65+c$

$F(1)=2(1)^2+3(1)+c=5+c$

$F(5)-F(1)=65-5$

$=60$

Summary:

The area under the curve (between the curve and the x-axis) on the interval [a,b] can be represented by the Fundamental Theorem of Calculus:

$$A=\int_a^b f(x)dx=F(x)\Big|_a^b=F(b)-F(a)$$

$$A=\int_1^5 f(x)dx=F(b)-F(a)$$

When we find area we are also evaluating definite integrals.

Examples – Page 455: 1kl, 2c. Page 461 : 1k

1. Find the area under the curve from *a* to *b*.

k) $y=x^2-x+2$, from -2 to 1.

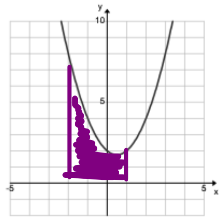
CALC

$y_1=x^2-x+2$

7. $\int f(x)\,dx$

lower $x=-2$

upper $x=1$



$$A=\int_{-2}^{1}(x^2-x+2)dx=\frac{1}{3}x^3-\frac{1}{2}x^2+2x\Big|_{-2}^{1}$$

$A=F(1)-F(-2)$

$=\left[\frac{1}{3}(1)^3-\frac{1}{2}(1)^2+2(1)\right]-\left[\frac{1}{3}(-2)^3-\frac{1}{2}(-2)^2+2(-2)\right]$

$=\left[\frac{1}{3}-\frac{1}{2}+2\right]-\left[-\frac{8}{3}-2-4\right]$

$=\frac{1}{3}-\frac{1}{2}+2+\frac{8}{3}+6$

$=3+8-\frac{1}{2}$

$=10.5$

$=\boxed{\frac{21}{2}}$

4. Area

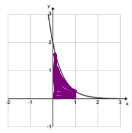
l) $y = 2e^{-2x}$, from 0 to 1.

$A = \int_0^1 2e^{-2x}\,dx$

divide by $\frac{d}{dx}(-2x) = -2$

$= \frac{2e^{-2x}}{-2}$

$= -1e^{-2x} \Big|_0^1$



$A = F(1) - F(0)$

$= \left[-1e^{-2(1)}\right] - \left[-1e^{-2(0)}\right]$

$= -\frac{1}{e^2} + 1\left(\frac{e^2}{e^2}\right) = \frac{e^2 - 1}{e^2}$

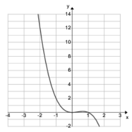
2. Find the area below the curve and above the x-axis:

c) $y = x^2 - x^3$, from -2 to 1

$A = \int_{-2}^{1} (x^2 - x^3)\,dx$

$A = \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_{-2}^{1}$

$A = F(1) - F(-2)$



$A = \left[\frac{1}{3}(1)^3 - \frac{1}{4}(1)^4\right] - \left[\frac{1}{3}(-2)^3 - \frac{1}{4}(-2)^4\right]$

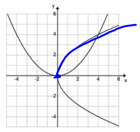
$A = \frac{1}{3} - \frac{1}{4} + \frac{8}{3} + \frac{16}{4} = 3 + \frac{15}{4} = \frac{12}{4} + \frac{15}{4}$

$A = \frac{27}{4}$

Page 461 :

1. Sketch the region bounded by the given curves and find the area of the region.

k) $y^2 = 4x$ and $x^2 = 4y$



top $y^2 = 4x$

$A = \int$ top function - bottom function

OR

$A = \int \text{top} - \int \text{bottom}$

I. intervals from intersection pts

solve the system

$4y = x^2$

$y = \frac{1}{4}x^2$ $\qquad y^2 = 4x$

$\left[\frac{1}{4}x^2\right]^2 = 4x$

$\frac{1}{16}x^4 = 4x$

$\frac{1}{16}x^4 - 4x = 0$

$\frac{1}{16}x\left[x^3 - 64\right] = 0$

$\frac{1}{16}x = 0 \qquad x^3 - 64 = 0$ or $(x-4)(x^2 + 4x + 16) = 0$

$x = 0 \qquad x^3 = 64$

$x = 4$

II. top function:

$y = \sqrt{4x}$

$y = 2\sqrt{x}$

bottom:

$y = \frac{1}{4}x^2$

III.

$A = \int_0^4 \left(2\sqrt{x} - \frac{1}{4}x^2\right) dx$ OR $A = \int_0^4 2\sqrt{x} - \int_0^4 \frac{1}{4}x^2$

or $2x^{1/2}$

$A = \frac{2x^{3/2}}{3/2} - \frac{1}{4}\left(\frac{x^3}{3}\right) \Big|_0^4$

$A = \frac{4}{3}x^{3/2} - \frac{1}{12}x^3 \Big|_0^4$

$A = F(4) - F(0)$

$A = \left[\frac{4}{3}(4)^{3/2} - \frac{1}{12}(4)^3\right] - \left[0\right]$

$= \frac{4}{3}(8) - \frac{16}{3}$

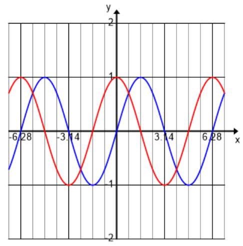
$= \frac{16}{3}$

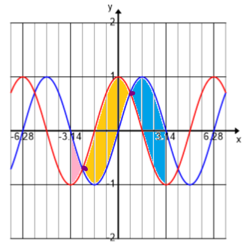
4. Area

EXERCISE 10.2

B **1.** Find the area of the region between the given curves. Include a sketch of the region.

(o) $y = \sin x$ and $y = \cos x$ from $-\pi$ to π





$\boxed{y = \sin x}$ and $\boxed{y = \cos x}$ intersect

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$ $\quad (-\pi, \pi)$

$x = \frac{\pi}{4} \quad x = -\frac{3\pi}{4}$

THREE REGIONS on top-bottom

$A_1 = \int_{-\pi}^{-\frac{3\pi}{4}} (\sin x - \cos x)\,dx \qquad A_2 = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (\cos x - \sin x)\,dx \qquad A_3 = \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,dx$

$A_1 = -\cos x - \sin x \Big|_{-\pi}^{-\frac{3\pi}{4}}$

$= F(-\frac{3\pi}{4}) - F(-\pi)$

$= \left[-\cos(\frac{3\pi}{4}) - \sin(\frac{-3\pi}{4})\right] - \left[-\cos(-\pi) - \sin(-\pi)\right]$

$= \left[-\frac{-\sqrt{2}}{2} - \frac{-\sqrt{2}}{2}\right] - \left[-(-1) - 0\right]$

$= \sqrt{2} - 1$

$A_2 = \sin x + \cos x \Big|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}$

$= F(\frac{\pi}{4}) - F(-\frac{3\pi}{4})$

$= \left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right] - \left[\cos\frac{-3\pi}{4} + \sin\frac{-3\pi}{4}\right]$

$= \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] - \left[\frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}\right]$

$= \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2}$

$A_3 = -\cos x - \sin x \Big|_{\frac{\pi}{4}}^{\pi}$

$= F(\pi) - F(\frac{\pi}{4})$

$= \left[-\cos(\pi) - \sin(\pi)\right] - \left[-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right]$

$= \left[-(-1) - (0)\right] - \left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right]$

$= 1 + \sqrt{2}$

$A = \left[\sqrt{2} - 1 + 2\sqrt{2} + 1 + \sqrt{2}\right]$

$A = 4\sqrt{2}$