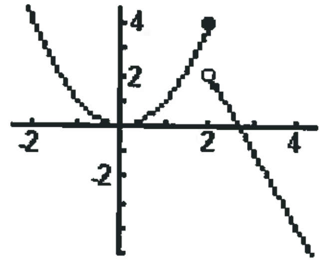
MATH 31

Name Key.

FINAL EXAM

1. Given the graph of g(x).



a) Find the following limits, if they exist.

i. $\lim_{x\to 2^-} g(x)$ $=4$

ii. $\lim_{x\to 2^+} g(x)$ $=2$

iii. $\lim_{x\to 2} g(x)$ $=$ U.

4

b) Why is the function g(x) discontinuous? Justify by identifying the condition of continuity not met.

$\lim_{x\to 2} g(x)$ is not defined $\therefore$ g(x) is discontinuous.

2. Find the limit of

a) $\lim_{x\to 2} \frac{x^2+2x-8}{x^2-4} = \lim_{x\to 2} \frac{(x+4)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2} \frac{x+4}{x+2}$

$= \frac{6}{4}$

$= \frac{3}{2}$

4

b) $\lim_{n\to\infty} \frac{6n^2+3n+5}{2n^2+5} \left(\frac{1/n^2}{1/n^2}\right)$

$= \lim_{n\to\infty} \frac{6+\frac{3}{n}+\frac{5}{n^2}}{2+\frac{5}{n^2}} = \frac{6}{2} = 3$

8

3. The point P(2,-8) lies on the parabola $y = x^2 - 6x$

a) Find the slope of the tangent line to the parabola at P. Use first principles.

$$m = \lim_{h \to 0} \frac{f(h+a) - f(a)}{h}$$

$$m = \lim_{h \to 0} \frac{[(x+h)^2 - 6(x+h)] - [x^2 - 6x]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 6x - 6h - x^2 + 6x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 6h}{h}$$

$$= \lim_{h \to 0} 2x + h - 6 = 2x - 6$$

4

b) Find the equation of the tangent line at P.

$\therefore$ slope at $P = 2(2) - 6$
$= -2$

$y + 8 = -2(x-2)$

$y + 8 = -2x + 4$

$2x + y + 4 = 0$ OR $y = -2x - 4$

4. Find $\frac{dy}{dx}$ for each of the following.

a) $y = 8x^3 - 5x^2 + 10\sqrt{x^3} - 75$

$y = 8x^3 - 5x^2 + 10x^{3/2} - 75$

$y' = 24x^2 - 10x + 15x^{1/2}$

2

b) $y = (3x-5)^4(x^2+9)$

$y' = \{4(3x-5)^3(3)\}(x^2+9) + (3x-5)^4(2x)$

3

$y' = 2(3x-5)^3[2(3)(x^2+9) + x(3x-5)]$

$y' = 2(3x-5)^3(6x^2 + 54 + 3x^2 - 5x)$

$y' = 2(3x-5)^3(9x^2 - 5x + 54)$

/9

c) $y = \dfrac{x}{\sqrt{1+2x}}$

$$y' = \frac{1(1+2x)^{1/2} - x\left(\tfrac{1}{2}(1+2x)^{-1/2}(2)\right)}{1+2x}$$

$$y' = \frac{(1+2x)^{-1/2}\left[1(1+2x) - x\right]}{1+2x}$$

$$y' = \frac{x+1}{(1+2x)^{3/2}}$$

d) $2xy = x^2 + y^2$

$$2y + 2xy' = 2x + 2yy'$$

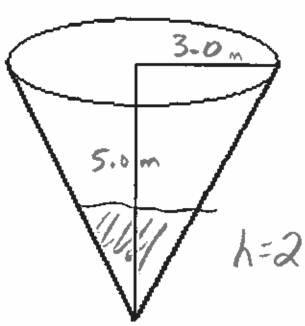
$$2xy' - 2yy' = 2x - 2y$$

$$y' = \frac{2(x-y)}{2(x-y)}$$

$$y' = 1$$

5. A water tank is built in the shape of a right circular cone with a height of 5.0 m and a diameter of 6.0 m at the top. Water is being pumped into the tank at a rate of 1.6 m^3/min. Find the rate at which the water level is rising when the water is 2.0 m deep?

[$V = \frac{1}{3}\pi r^2 h$ Round your answer to the nearest hundredth m/min]



3

$$\frac{dV}{dt} = 1.6 \text{ m}^3/\text{min}$$

$$V = \frac{1}{3}\pi\left(\frac{3h}{5}\right)^2(h)$$

$$\frac{r}{h} = \frac{3}{5}$$

$$r = \frac{3h}{5}$$

$$\frac{d}{dt}\left[V = \frac{3\pi}{25}h^3\right]$$

$$\frac{dV}{dt} = \frac{9\pi}{25}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1.6 \times 25}{9\pi} \times \frac{1}{2^2}$$

$$= \frac{10}{9\pi}$$

$$\approx 0.35 \text{ m/min}$$

MATH 31 - FINAL EXAM

6. Find the equations for the vertical and horizontal asymptotes of the curve

$$y = \frac{2x^2 - 9}{x^2 - 9}$$

vertical

$x^2 = 9$

$x = \pm 3$

horizontal

$$y = \lim_{x \to \infty} \frac{2 - \frac{9}{x^2}}{1 - \frac{9}{x^2}} = \frac{2}{1}$$

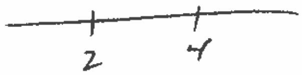
$y = 2$

3

7. Given the function: $f(x) = x^3 - 9x^2 + 24x - 20$

a) Find the interval(s) on which the function, f(x), is increasing and decreasing on.

$$f'(x) = 3x^2 - 18x + 24$$
$$= 3(x^2 - 6x + 8)$$
$$= 3(x-4)(x-2)$$



	$x-4$	$x-2$	$f'(x)$	$f(x)$
$(-\infty, 2)$	−	−	+	increasing
$(2, 4)$	−	+	−	decreasing
$(4, \infty)$	+	+	+	increasing

increasing $(-\infty, 2) \cup (4, \infty)$

decreasing $(2, 4)$

[4]

b) Find the regions of concavity and the points of inflection.

$$f''(x) = 6x - 18$$

	$6x-18$	$f''(x)$	$f(x)$
$(-\infty, 3)$	−	neg	CD
$(3, \infty)$	+	pos	CU

PI, $x = 3$

$$f(3) = (3)^3 - 9(3)^2 + 24(3) - 20$$
$$= 27 - 81 + 72 - 20$$
$$= -2$$

$\therefore$ PI $(3, -2)$

CD, $(-\infty, 3)$

CU $(3, \infty)$

… number 7 continues on the next page

7. continued... $f(x)=x^3-9x^2+24x-20$

c) Find the local maximum(s) and minimum(s) points. Justify your answer.

$f'(x)=0$

$0=3(x-2)(x-4)$

$x=2 \quad x=4$

$f''(2)=-6 \therefore$ CD, local max

$f''(4)=6 \therefore$ CU, local min

$f(2)=0$, then $(2,0)$ local max

$f(4)=-4$, then $(4,-4)$ local min

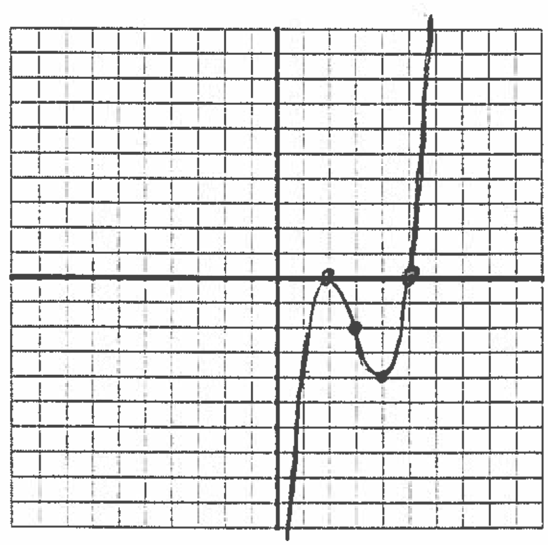
[4]

d) Algebraically find the zeros for $f(x)=x^3-9x^2+24x-20$ and sketch.

$$\begin{array}{r|rrrr} 2 & 1 & -9 & 24 & -20 \\ & & 2 & -14 & 20 \\ \hline 2 & 1 & -7 & 10 & \boxed{0} \\ & & 2 & -10 & \\ \hline & 1 & -5 & \boxed{0} & \end{array}$$

$(x-2)(x-2)(x-5)=f(x)$

zeros are 2 and 5.



8. Evaluate $\lim_{x\to 0}\frac{1-\cos x}{2x^2}\left(\frac{1+\cos x}{1+\cos x}\right) = \lim_{x\to 0}\frac{1-\cos^2 x}{2x^2(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\lim_{x\to 0}\frac{1}{2(1+\cos x)}$

[2] $= (1)^2\left(\frac{1}{2(1+1)}\right)$

$= \frac{1}{4}$

9. Find the slope of $y = \sin 2x - \cos x$ when $x = \frac{\pi}{4}$.

$m = \frac{dy}{dx} = (\cos 2x)(2) + \sin x$

[3] $= \cos\left(2\cdot\frac{\pi}{4}\right)(2) + \sin\left(\frac{\pi}{4}\right)$

$= (0)(2) + \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{2}$

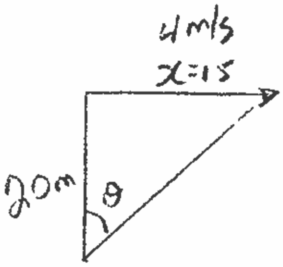
10. Find $\frac{dy}{dx}$ given $\left[\tan 2x = \cos 3y\right]\frac{d}{dx}$

$(\sec^2 2x)(2) = (-\sin 3y)(3y')$

[3] $y' = -\frac{2\sec^2 2x}{3\sin 3y}$

11 ~~10~~. A vehicle moves along a straight path with a speed of 4.0 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the vehicle. At what rate is the searchlight rotating when the vehicle is 15 m from the point on the path closest to the searchlight? (Rounded to the nearest hundredth radian/second.)

$\boxed{14}$



$\tan\theta = \frac{x}{20}$

$x = 20\tan\theta$

$\frac{dx}{dt} = 20\sec^2\theta \, \frac{d\theta}{dt}$

$\tan\theta = \frac{15}{20}$

$\theta = 36.9^\circ$ OR 0.64

$\frac{dx}{dt} = 4.0$

$\frac{4.0}{20} \times \cos^2(0.64) = \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = 0.13 \text{ rad/s}$

(12) ~~11~~. Differentiate with respect to x.

a) $y = e^{\tan\sqrt{x}}$

$$\frac{dy}{dx} = e^{\tan\sqrt{x}}\left[\left(\sec^2\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right)\right]$$

$$= e^{\tan\sqrt{x}}\left(\frac{\sec^2\sqrt{x}}{2\sqrt{x}}\right) \quad \text{OR} \quad \frac{e^{\tan\sqrt{x}}}{2\sqrt{x}\cos^2\sqrt{x}}$$

[4]

b) $y = \dfrac{e^{4x}}{x^2+1}$

$$\frac{dy}{dx} = \frac{\left[e^{4x}(4)\right](x^2+1) - e^{4x}(2x)}{(x^2+1)^2}$$

$$= \frac{2e^{4x}\left[2(x^2+1) - x\right]}{(x^2+1)^2}$$

$$= \frac{2e^{4x}(2x^2 - x + 2)}{(x^2+1)^2}$$

13. ~~12~~. Find the indefinite integrals.

a) $\int (3x^5 - \frac{1}{x})dx$ $= 3\left(\frac{x^6}{6}\right) - \ln|x| + c$

$= \frac{1}{2}x^6 - \ln|x| + c$

[5]

b) $\int 24x^2(4x^3+1)^4\,dx$

$= 24x^2\left(\frac{(4x^3+1)^5}{5}\right)\left(\frac{1}{12x^2}\right) + c$

$= \frac{2}{5}(4x^3+1)^5 + c$

14 ~~12~~. Find the function F given that (2,3) is on the graph $y = F(x)$ and, $F'(x) = 3x^2 - 2x + 6$.

$\int 3x^2 - 2x + 6 \, dx = x^3 - x^2 + 6x + C$ ... $y = 3$ when $x = 2$

$\boxed{3}$

$3 = 8 - 4 + 12 + C$

$\therefore \; F(x) = x^3 - x^2 + 6x - 13$

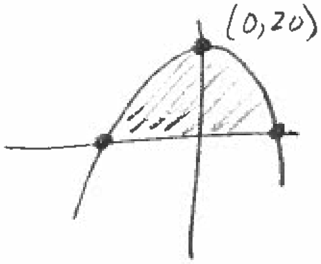
(15) ~~13~~. Find the exact area bounded by the curve $y = 20 - 5x^2$ and the x-axis. Include a sketch.

$20 - 5x^2 = 0$

$x^2 = 4$

$x = \pm 2$

[3]



$$A = \int_{-2}^{2} (20 - 5x^2)\, dx$$

$$= 20x - \frac{5}{3}x^3 \Big]_{-2}^{2}$$

$$= \left[20(2) - \frac{5}{3}(2)^3\right] - \left[20(-2) - \frac{5}{3}(-2)^3\right]$$

$$= 40 - \frac{40}{3} + 40 - \frac{40}{3} = \frac{160}{3} \text{ units}^2$$

(16) ~~14~~. Find the interval of intersection and the area between the curves: $y = 4 - x^2$ and $y = 2x^2 - 8$. Include a sketch.

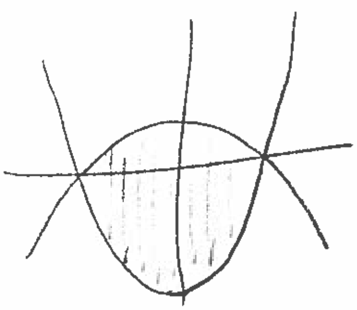
$4 - x^2 = 2x^2 - 8$

$12 = 3x^2$

$4 = x^2$

$x = \pm 2$

[3]



$$A = \int_{-2}^{2} (4 - x^2) - (2x^2 - 8)\, dx$$

$$= \int_{-2}^{2} 12 - 3x^2\, dx$$

$$= 12x - x^3 \Big]_{-2}^{2}$$

$$= \left[12(2) - (2)^3\right] - \left[12(-2) - (-2)^3\right]$$

$$= 24 - 8 + 24 - 8$$

$$= 32 \text{ units}^2$$