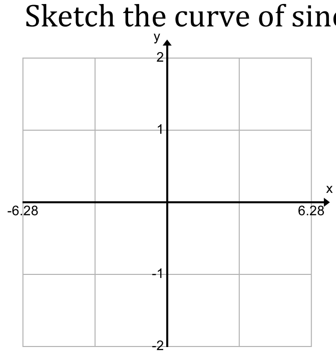
The Limits of Trigonometric Expressions

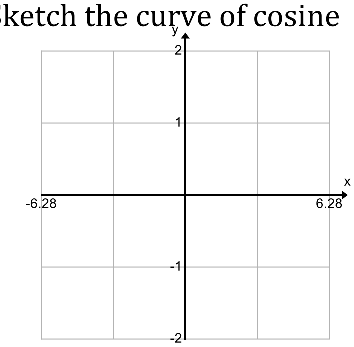
Review Limits (evaluate):

$\lim\limits_{x \to 5} \sqrt{x+4}$ $\lim\limits_{x \to 5} \frac{x^2-6x+5}{x^2-25}$

Outcomes: Find the limits of sine and cosine and simple modifications to them.

Warm up: Sketch the curve of sine Sketch the curve of cosine





Investigate: Look at the left- and right-hand limits of $\sin\theta$ and $\cos\theta$ as $x \to 0$
Since we know that sine and cosine are continuous graphs state the value of $\lim\limits_{\theta \to 0} \sin\theta$ and $\lim\limits_{\theta \to 0} \cos\theta$.

Examples:

1. Evaluate limits, not divide by zero.

a) $\lim\limits_{\theta \to \pi} \frac{\sin\theta}{2}$ b) $\lim\limits_{x \to 0} (\sin x + x)$. c) $\lim\limits_{x \to \pi} (\sin x + \cos x)$

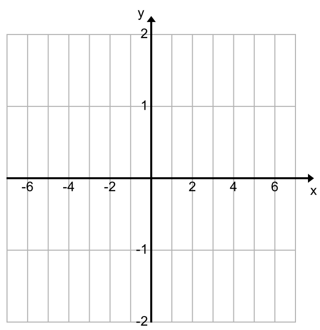
d) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sin x + 1}{\cos x + 1}$ e) $\lim\limits_{x \to \frac{3\pi}{2}} \frac{\cos x}{2x}$ f) $\lim\limits_{x \to 0} \frac{\cos 2x}{3\cos 3x}$

Investigate: Some limits which will be very important to trigonometric functions are:

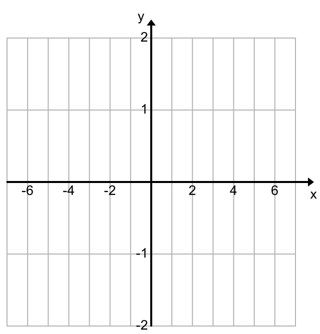
$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \quad \text{or} \quad \lim_{\theta \to 0} \frac{\theta}{\sin\theta} \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$$

These limits are necessary in order to find the derivatives of trigonometric functions.

Graph: $y = \frac{\sin\theta}{\theta}$ What does the graph indicate the $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ is equal to?



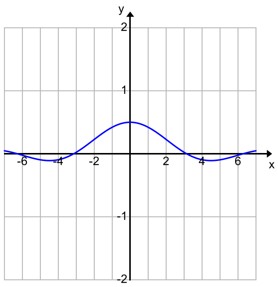
Graph: $y = \frac{\cos\theta - 1}{\theta}$ What does the graph indicate the $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta}$ is equal to?



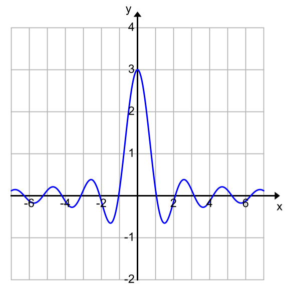
Examples:

2. Evaluate limits, dividing by zero: Graphically to see Algebraically to justify when $\left(\frac{0}{0}\right)$.

a) $\lim_{\theta \to 0} \frac{\sin \theta}{2\theta}$



b) $\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}$

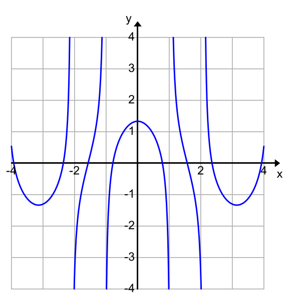


c) $\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta}$

d) $\lim_{x \to 0} x \sec x$.

e) $\lim_{x \to 0} \frac{\tan x}{\sin x}$.

f) $\lim_{x \to 0} \frac{\sin 4x}{\sin 3x}$



g) $\lim_{\theta \to \frac{\pi}{2}} \frac{\sin \theta}{2\theta}$

h) $\lim_{x \to 0} \frac{\sin^2 3x}{\sin^2 4x}$

i) $\lim_{x \to \frac{\pi}{4}} \frac{\cos 2x}{\cos x - \sin x}$

j) $\lim_{x \to 0} \frac{\tan x}{\tan 2x}$

Homework: Page 306: 1,2,7,9, 11,12,13,15,16,17,18,19,20,21,23,27,31,33

Derivatives of the Primary Trigonometric Ratios

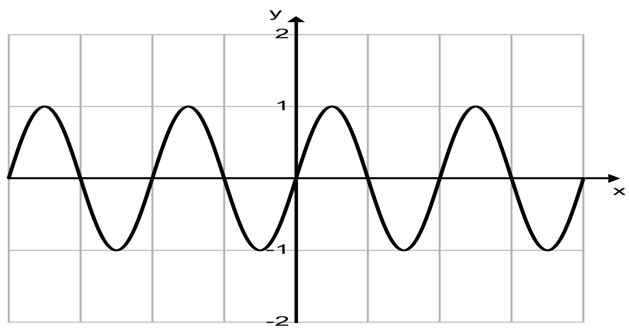
Given: $\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Outcomes: Find derivatives of sine and cosine and simple modifications to them.

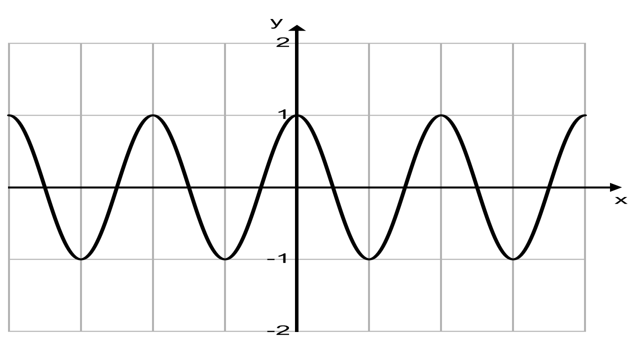
Recall: the derivative of a function was the slope of the tangent to a curve and that $\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$. This is the definition of derivative using the concept of first principles.

1. Determine the derivative of $y = \sin(x)$
    - Using first principles.

    - Given the curve of $y = \sin(x)$. Use the concept of tangent line slopes to graphically find the derivative of $y = \sin(x)$.



2. Determine the derivative of $y = \cos(x)$.
    - Given the curve of $y = \cos(x)$. Use the concept of tangent line slopes to graphically find the derivative of $y = \cos(x)$.



If $y = \sin u$ or $y = \cos u$ is a composition of two functions, use the chain rule:

- $\frac{d}{dx} \sin u = \cos u \times \frac{du}{dx}$
- $\frac{d}{dx} \cos u = -\sin u \times \frac{du}{dx}$

Examples:

1. Find the derivative of the following

a) $y = \sin 2x$

b) $y = \sin(x^2 - 1)$

c) $y = x \sin x$

d) $y = \dfrac{x}{\sin x}$

e) $y = \cos^2 x$

f) $y = \cos(ax + b)$

g) $y = \cos(\sin x)$

h) Differentiate implicitly $\sin x + \sin y = 1$

2. Find the equation of the tangent line to $y = \dfrac{\sin x}{\cos 2x}$ at $x = \dfrac{\pi}{6}$

Homework:

Calculus, A First Course. McGraw-Hill Ryerson Limited, 1989.
Pg 313# 1 (second column), 2, 3(a,e), 4(a), 5(b), 11(a,c)

B 1. Find the derivative of y with respect to x in each of the following.

(a) $y = \cos(-4x)$

(b) $y = \sin(3x + 2\pi)$

(c) $y = 4\sin(-2x^2 - 3)$

(d) $y = -\frac{1}{2}\cos(4 + 2x)$

(e) $y = \sin x^2$

(f) $y = -\cos x^2$

(g) $y = \sin^{-2}(x^3)$

(h) $y = \cos(x^2 - 2)^2$

(i) $y = 3\sin^4(2 - x)^{-1}$

(j) $y = x\cos x$

(k) $y = \dfrac{x}{\sin x}$

(l) $y = \dfrac{\sin x}{1 + \cos x}$

(m) $y = (1 + \cos^2 x)^6$

(n) $y = \sin \dfrac{1}{x}$

(o) $y = \sin(\cos x)$

(p) $y = \cos^3(\sin x)$

(q) $y = x \cos \dfrac{1}{x}$

(r) $y = \dfrac{\sin^2 x}{\cos x}$

(s) $y = \dfrac{1 + \sin x}{1 - \sin 2x}$

(t) $y = \sin^3 x + \cos^3 x$

(u) $y = \cos^2\left(\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)$

2. Find $\frac{dy}{dx}$ in each of the following.

(a) $\sin y = \cos 2x$ (b) $x \cos y = \sin(x + y)$

(c) $\sin y + y = \cos x + x$ (d) $\sin(\cos x) = \cos(\sin y)$

(e) $\sin x \cos y + \cos x \sin y = 1$

(f) $\sin x + \cos 2x = 2xy$

3. Find an equation of the tangent line to the given curve at the given point.

(a) $y = 2 \sin x$ at $\left(\frac{\pi}{6}, 1\right)$ (b) $y = \frac{\sin x}{\cos x}$ at $\left(\frac{\pi}{4}, 1\right)$

(c) $y = \frac{1}{\cos x} - 2 \cos x$ at $\left(\frac{\pi}{3}, 1\right)$

(d) $y = \frac{\cos^2 x}{\sin^2 x}$ at $\left(\frac{\pi}{4}, 1\right)$

(e) $y = \sin x + \cos 2x$ at $\left(\frac{\pi}{6}, 1\right)$

(f) $y = \cos(\cos x)$ at $x = \frac{\pi}{2}$

4. Find the critical numbers, the intervals of increase and decrease, and any maximum or minimum values.

(a) $y = \sin^2 x, \ -\pi \leqslant x \leqslant \pi$

(b) $y = \cos x - \sin x, \ -\pi \leqslant x \leqslant \pi$

5. Determine the concavity and find the points of inflection.

(a) $y = 2 \cos x + \sin 2x, \ 0 \leqslant x \leqslant 2\pi$

(b) $y = 4 \sin^2 x - 1, \ -\pi \leqslant x \leqslant \pi$

11. Find $\frac{dy}{dx}$ in each of the following.

(a) $y = \frac{1}{\sin(x - \sin x)}$ (b) $y = \sqrt{\sin\sqrt{x}}$

(c) $y = \sqrt[3]{x \cos x}$

(d) $y = \cos^3(\cos x) + \sin^2(\cos x)$

(e) $y = \sqrt{\cos(\sin^2 x)}$

Derivatives of the Other Trigonometric Ratios

Outcomes: Find derivatives of tangent, cosecant, secant and cotangent

Investigate:

1. Write tan *x* in terms of sine and cosine. Use the quotient rule to find the derivative of tan *x*.

2. Write sec *x* in terms of cosine Find the derivative of sec *x*.

3. Write csc *x* terms of sine. Find the derivative of csc x.

4. Write cotan *x* in sine and cosine. Use the quotient rule to find the derivative of cotan *x*.

SUMMARY:

$$\frac{d}{dx}\tan u = \sec^2 u \times \frac{du}{dx} \qquad \frac{d}{dx}\sec u = \sec u \times \tan u \times \frac{du}{dx}$$

$$\frac{d}{dx}\csc u = -\csc u \times \cot u \times \frac{du}{dx} \qquad \frac{d}{dx}\cot u = -\csc^2 u \times \frac{du}{dx}$$

Use previous rules and function properties to differentiate etc

1. Find the derivative of the following
   a) $y = x^2 \tan x$
   b) $y = \sin x + \tan x$
   c) $y = \dfrac{\tan^2 x}{x}$
   d) $y = \sqrt{\tan 2x}$
   e) $y = \dfrac{1}{\tan x + 1}$
   f) $y = \sec\left(x^2 + 1\right)$
   g) $y = \cot\sqrt{x}$.
   h) $y = 2\csc 3x$.
   i) $y = 2\sec^2\left(2x^3\right)$

2. Find the vertical asymptotes of $y = \sec x + \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Calculus, A First Course. McGraw-Hill Ryerson Limited, 1989.
Pg 319 #1(first column), 2(a,e), 3(b,c), 7, 9(a)

1. Find the derivative of each of the following.

(a) $y = 3 \tan 2x$
(b) $y = \frac{1}{3} \cot 9x$
(c) $y = 12 \sec \frac{1}{4}x$
(d) $y = -\frac{1}{4} \csc(-8x)$
(e) $y = \tan x^2$
(f) $y = \tan^2 x$
(g) $y = \sec \sqrt[3]{x}$
(h) $y = x^2 \csc x$
(i) $y = \cot^3(1 - 2x)^2$
(j) $y = \sec^2 x - \tan^2 x$
(k) $y = \dfrac{1}{\sqrt{(\sec 2x - 1)^3}}$
(l) $y = \dfrac{x^2 \tan x}{\sec x}$
(m) $y = 2x(\sqrt{x} - \cot x)$
(n) $y = \sin(\tan x)$
(o) $y = \tan^2(\cos x)$
(p) $y = [\tan(x^2 - x)^{-2}]^{-3}$

2. Find $\dfrac{dy}{dx}$.

(a) $\tan x + \sec y - y = 0$
(b) $\tan 2x = \cos 3y$
(c) $\cot(x + y) + \cot x + \cot y = 0$
(d) $y^2 - \csc(xy) = 0$
(e) $x^2 + \sec\left(\dfrac{x}{y}\right) = 0$
(f) $y^2 = \sin(\tan y) + x^2$

3. Find the equations of the tangent lines.

(a) $y = \cot^2 x$ when $x = \dfrac{\pi}{4}$
(b) $y = \sin x \tan \dfrac{x}{2}$ when $x = \dfrac{\pi}{3}$
(c) $y = \csc 2x$ when $x = -\dfrac{\pi}{8}$

Trigonometric Problem Solving

Objectives: Solve applications of trigonometric function questions

Problem Solving:

- Draw a diagram, label things that change with variables.
- Match the number of variables to the number of rates in the problem.
- Determine which trig ratio to work with – you may have two choices, select the easiest one to derive.
- Put rates into the question – it's usually time: $^{d}/_{dt}$
- Solve for the moment – we usually need to solve for one piece of the puzzle on our own.

1. The beam of a lighthouse sweeps across the path of a boat cruising at a speed of 30 km/h parallel to the shoreline. If the boat is 2 km from the shore and stays within the beam of the light, at what rate is the beam revolving (in rad/h) when the boat has sailed 4 km from a point opposite the lighthouse.

2. Two sides of a triangle have lengths of 15m and 20m. The angle between them is increasing at $\frac{\pi}{90}{}^{rad}/_{s}$. How fast is the length of the third side changing when the angle between the sides is $\frac{\pi}{3}$?

3. The angle of elevation of the sun is decreasing at $\frac{1}{3}$ rad/h. How fast is the shadow cast by a tree 10 m tall lengthening when the angle of elevation of the sun is $\frac{\pi}{3}$ rad?

4. A ladder 8 m long is resting against the vertical wall of a house. If the top of the ladder is sliding down the wall and the angle the ladder makes with the ground is decreasing at a rate $\frac{1}{4}$ of rad/s, how fast is the ladder sliding down the wall, when the angle is $\frac{\pi}{4}$ rad?

5. Find the maximum perimeter of a right triangle with hypotenuse 20 cm.

6. An airplane, in level flight, is approaching the spot where you are standing. The speed of the airplane is 100 m/s and it is flying at an altitude of 1000 m. What is the rate of change of the angle of elevation q when the distance from where you are standing to a point directly below the plane is 2000 m?

7. A video camera at ground level is filming the liftoff of a hot-air balloon that is rising vertically according to the position equation $h = 2t$, where h is in metres and t is in seconds. If the camera is 100 m from the launch site, find the rate of change of the angle of elevation of the camera 5 s after liftoff.

Page 325 # 1a, 2b, 8, 10, 11, 12, 14

1. Find the local maxima and/or minima of each of the following functions.

(a) $f(x) = x - 2 \sin x, 0 \leqslant x \leqslant 2\pi$

(b) $f(x) = x + \cos x, 0 \leqslant x \leqslant 2\pi$

(c) $f(x) = \sin^4 x + \cos^4 x, 0 \leqslant x \leqslant 2\pi$

(d) $f(x) = x \sin x + \cos x, -\pi \leqslant x \leqslant \pi$

2. The position of a particle as it moves horizontally is described by the given equations. If s is the displacement in metres and t is the time in seconds find the absolute maximum and absolute minimum displacements.

(a) $s = 2 \sin t + \sin 2t, -\pi \leqslant t \leqslant \pi$

(b) $f(t) = \sin^2 t - 2 \cos^2 t, -\pi \leqslant t \leqslant \pi$

8. The angle of elevation of the sun is decreasing at $\frac{1}{4}$ rad/h. How fast is the shadow cast by a building of height 50 m lengthening, when the angle of elevation of the sun is $\frac{\pi}{4}$?

10. A revolving beacon is situated 925 m from a straight shore. It turns at 2 rev/min. How fast does the beam sweep along the shore at its nearest point? How fast does it sweep along the shore at a point 1275 m from the nearest point?

11. Two sides of a triangle are six and eight metres in length. If the angle between them decreases at the rate of 0.035 rad/s, find the rate at which the area is decreasing when the angle between the sides of fixed length is $\frac{\pi}{6}$.

12. A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 m/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$?

14. A vehicle moves along a straight path with a speed of 4 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the vehicle. At what rate (in rad/s) is the searchlight rotating when the vehicle is 15 m from the point on the path closest to the searchlight?